1. (a) A homomorphism from the vector space $\mathbb{R}^3$ to the set of traceless hermitian $2 \times 2$ matrices is defined by $\vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}$. First, show that $\det(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}) = -|\vec{\boldsymbol{x}}|^2$. Second, prove the identity:

$$x_i = \tfrac{1}{2}\mathrm{Tr}\,(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,\sigma_i)\,.$$

This identity provides the inverse transformation from the set of traceless $2 \times 2$ hermitian matrices to the vector space $\mathbb{R}^3$.

Explicitly, we have

$$\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}\,.$$

It follows that

$$\det \vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} = -x_1^2 - x_2^2 - x_3^2 = -|\vec{\boldsymbol{x}}|^2\,. \tag{1}$$

To invert the transformation $\vec{\boldsymbol{x}} \longrightarrow \vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}$, note that:

$$\mathrm{Tr}\,\left[(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})\sigma_i\right] = x_j\mathrm{Tr}(\sigma_j\sigma_i) = 2x_i\,,$$

after using $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$. Hence, it follows that

$$x_i = \tfrac{1}{2}\mathrm{Tr}\left[(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})\sigma_i\right]\,.$$

(b) Let $U \in \mathrm{SU}(2)$. Show that $U\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1} = \vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}$ for some vector $\vec{\boldsymbol{y}}$. Using the results of part (a), prove that an element of the rotation group exists such that $\vec{\boldsymbol{y}} = R\vec{\boldsymbol{x}}$ and determine an explicit form for $R \in \mathrm{SO}(3)$. Display a homomorphism from SU(2) onto SO(3) and prove that SO(3)$\cong$ SU(2)/$\mathbb{Z}_2$.

We can evaluate $U\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1}$ by writing $U = \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2)$ and using the results of part (c) of problem 7 of Solution Set 2. It follows that:

$$U\sigma_j U^{-1} = e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} = R_{ij}(\hat{\boldsymbol{n}},\theta)\,\sigma_i\,,$$

where $R_{ij}$ is given by

$$R_{ij}(\hat{\boldsymbol{n}},\theta) = n_i n_j + (\delta_{ij} - n_i n_j)\cos\theta - \epsilon_{ijk}n_k\sin\theta\,. \tag{2}$$

Thus,

$$U\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1} = x_j R_{ij}\sigma_i = \vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,, \tag{3}$$

where $y_i \equiv R_{ij}x_j$. Thus, we have explicitly identified the matrix $R$ which is an orthogonal $3 \times 3$ matrix of unit determinant, i.e. $R \in \mathrm{SO}(3)$.

**Remarks**

The derivation of eq. (3) can be understood in a more general context. In class we showed that for a Lie group $G$, we can define the adjoint representation of $G$, $\{\mathrm{Ad}_g \mid g \in G\}$, where

$$\mathrm{Ad}_g(X) = gXg^{-1}\,, \quad \text{for } g \in G \text{ and } X \in \mathfrak{g}, \tag{4}$$

where $\mathfrak{g}$ is the Lie algebra of the Lie group $G$. The explicit matrices of the adjoint representation, $D(g)$, are obtained by the action of $\mathrm{Ad}_g$ on the basis vectors of the Lie algebra $\mathfrak{g}$,

$$\mathrm{Ad}_g(\mathcal{A}_j) = D(g)_{ij}\,\mathcal{A}_i\,, \tag{5}$$

where $[\mathcal{A}_i, \mathcal{A}_j] = f_{ij}^k \mathcal{A}_k$. Applying eqs. (4) and (5) to the defining representation of SU(2), we identify $g = U$, $\mathcal{A}_j = -\frac{1}{2}i\sigma_j$, $f_{ij}^k = \epsilon_{ijk}$ and $D(g)_{ij} = R_{ij}$, where $R$, whose explicit form is given by eq. (2), are the matrices of the adjoint representation of SU(2), which correspond to the $3 \times 3$ rotation matrices [i.e., the elements of the defining representation of SO(3)]. That is,

$$U\sigma_j U^{-1} = R_{ij}\sigma_i\,. \tag{6}$$

Multiplying both sides of eq. (6) by $x_j$ yields,

$$U\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1} = x_j R_{ij}\sigma_i = \vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,, \tag{7}$$

where $y_i = R_{ij}x_j$, thereby reproducing eq. (3).

**Alternative derivation**. To prove that a real vector $\vec{\boldsymbol{y}}$ exists such that

$$\vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} = U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1}\,,$$

for any real vector $\vec{\boldsymbol{x}}$, it is sufficient to prove that $U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1}$ is a $2 \times 2$ traceless hermitian matrix, since the most general $2 \times 2$ traceless hermitian matrix is always a real linear combination of the three Pauli matrices. First, we note that

$$\mathrm{Tr}(U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1}) = \mathrm{Tr}\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} = 0\,.$$

Next, since $U^\dagger = U^{-1}$ and $\vec{\boldsymbol{\sigma}}^\dagger = \vec{\boldsymbol{\sigma}}$, it follows that

$$(U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1})^\dagger = U(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^\dagger\,U^{-1} = U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1}\,,$$

where we have used the fact that the components of $\vec{\boldsymbol{x}}$ are real. Hence, $U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,U^{-1}$ is a $2 \times 2$ traceless hermitian matrix.

We now use the results of part (a) to conclude that:

$$y_j = \tfrac{1}{2}\mathrm{Tr}\big(\vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\big)\sigma_i = \tfrac{1}{2}\mathrm{Tr}(U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}U^{-1}\sigma_i) = \tfrac{1}{2}x_j\,\mathrm{Tr}(U\sigma_j U^{-1}\sigma_i) = R_{ij}x_j\,,$$

which yields an expression for $R_{ij}$ in terms of the SU(2) matrix $U$,

$$R_{ij} \equiv \tfrac{1}{2}\mathrm{Tr}(U\sigma_j U^{-1}\sigma_i)\,. \tag{8}$$

In fact, this definition coincides with eq. (50) of Solution Set 2, and we conclude that $R_{ij}$ is given explicitly by eq. (2). Indeed $R \in \mathrm{SO}(3)$ as expected.

However, one does not require an explicit calculation of eq. (8) to conclude that $R \in \mathrm{SO}(3)$. First, we note that

$$\det \vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} = \det(U\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}U^{-1}) = \det \vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,.$$

Using eq. (1), it follows that $|\vec{\boldsymbol{y}}|^2 = |\vec{\boldsymbol{x}}|^2$. Thus the linear transformation $\vec{\boldsymbol{y}} = R\vec{\boldsymbol{x}}$ preserve the length of the vector. This observation proves that $R \in \mathrm{O}(3)$. To show that $R \in \mathrm{SO}(3)$, we simply note that $R$ is continuously connected to the identity, corresponding to $U = \pm\mathbf{I}$. At this point, $R_{ij} = \frac{1}{2}\mathrm{Tr}(\sigma_j\sigma_i) = \delta_{ij}$, which implies that $\det R = 1$. Since SU(2) is a connected group, $\det R$ cannot change discontinuously as we vary $U \in \mathrm{SU}(2)$. Thus, we conclude that $\det R = 1$ for all $U \in \mathrm{SU}(2)$, which means that $R \in \mathrm{SO}(3)$ as expected.

The homomorphism of SU(2) onto SO(3) is easily obtained. It consists of the mapping:

$$\{U\,,\,-U\} \longmapsto R\,, \qquad \text{where } U \in \mathrm{SU}(2) \text{ and } R \in \mathrm{SO}(3)\,.$$

To prove that this is a homomorphism, we must show that the group multiplication law is preserved. If we define

$$U_2\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}U_2^{-1} = \vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} \quad \Longrightarrow \quad \vec{\boldsymbol{y}} = R(U_2)\vec{\boldsymbol{x}}\,, \tag{9}$$

$$U_1\vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}U_1^{-1} = \vec{\boldsymbol{z}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} \quad \Longrightarrow \quad \vec{\boldsymbol{z}} = R(U_2)\vec{\boldsymbol{y}}\,, \tag{10}$$

then

$$U_1U_2\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}(U_1U_2)^{-1} = \vec{\boldsymbol{z}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,, \quad \Longrightarrow \quad \vec{\boldsymbol{z}} = R(U_1U_2)\vec{\boldsymbol{x}}\,.$$

However, eqs. (9) and (10) yield $\vec{\boldsymbol{z}} = R(U_1)R(U_2)\vec{\boldsymbol{x}}$. Since these equations hold for arbitrary vectors, it follows that $R(U_1U_2) = R(U_1)R(U_2)$, which demonstrates that the group multiplication law is preserved.

From the explicit form for $R_{ij}$ given in eq. (8), it is clear that both $U$ and $-U$ correspond to the same rotation $R$. Equivalently, the kernel of the homomorphism defined by eq. (8) is $\mathbb{Z}_2 \cong \{\mathbf{I}\,,\,-\mathbf{I}\}$, as these are the only two SU(2) matrices that are mapped onto the identity element, $R_{ij} = \delta_{ij}$ of SO(3). Hence, the homomorphism defined by eq. (8) is a two-to-one mapping (also called a double-valued homomorphism). Hence, one can introduce an equivalence relation among SU(2) matrices such that $U_1 \sim U_2$ if either $U_1 = U_2$ or $U_1 = -U_2$. It therefore follows that

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2\,.$$

(c) The Lie group SU(1,1) is defined as the group of $2 \times 2$ matrices $V$ that satisfy $V\sigma_3V^\dagger = \sigma_3$ and $\det V = 1$. (Note that $V$ is *not* a unitary matrix.) The Lie group SO(2,1) is the group of transformations on vectors $\vec{\boldsymbol{x}} \in \mathbb{R}^3$ (with determinant equal to one) that preserves $x_1^2 + x_2^2 - x_3^2$. Display the homomorphism from SU(1,1) onto SO(2,1) and compare with part (b).

First we work out the Lie algebra of SU(1,1). Expanding $V$ about the identity, we write $V \simeq \mathbf{I} + W$. Then, using the definition of SU(1,1),

$$\sigma_3 \simeq (\mathbf{I} + W)\sigma_3(\mathbf{I} + W^\dagger)\,,$$

To first order, it follows that

$$W\sigma_3 = -\sigma_3 W^\dagger\,. \tag{11}$$

Moreover,

$$1 = \det(\mathbf{I}+W) \simeq 1 + \operatorname{Tr} W\,,$$

yields $\operatorname{Tr}W = 0$. The latter implies that $W$ is a complex linear combination of Pauli matrices,

$$W = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3\,.$$

If we now impose eq. (11), it follows that

$$(w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3)\sigma_3 = -\sigma_3(w_1^*\sigma_1 + w_2^*\sigma_2 + w_3^*\sigma_3)\sigma_3)\,.$$

Using $\sigma_i\sigma_j = \mathbf{I}\delta_{ij} + i\epsilon_{ijk}\sigma_k$, this condition simplifies to:

$$-iw_1\sigma_2 + iw_2\sigma_1 + w_3\mathbf{I} = -iw_1^*\sigma_2 + iw_2^*\sigma_1 - w_3^*\mathbf{I}\,.$$

That is, $w_1$ and $w_2$ are real and $w_3$ is pure imaginary. It is convenient to define three matrices,

$$\vec{\boldsymbol{\tau}} = (i\sigma_1\,,\,i\sigma_2\,,\,\sigma_3)\,.$$

Then, we may choose $\{-\frac{1}{2}i\tau_k\}$ as a basis for the real Lie algebra of SU(1 , 1). That is, the most general element of the real Lie algebra of SU(1 , 1) is

$$W = \sum_{k=1}^{3} c_k(-\tfrac{1}{2}i\tau_k)\,, \qquad \text{where } c_k \in \mathbb{R}\,,$$

where the factor of $\frac{1}{2}$ is conventional. The Lie group elements are obtained by exponentiation:

$$e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}}/2} = e^{\theta(n_1\sigma_1+n_2\sigma_2-in_3\sigma_3)/2} \in \ \mathrm{SU}(1\,,1)\,,$$

where $\hat{\boldsymbol{n}}$ is a unit vector.

We now repeat the analysis of parts (a) and (b). Note that

$$\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}} = \begin{pmatrix} x_3 & ix_1+x_2 \\ ix_1-x_2 & -x_3 \end{pmatrix}\,,$$

and

$$\det \vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}} = x_1^2 + x_2^2 - x_3^2\,.$$

If we define:

$$\varepsilon_k = \begin{cases} -1\,, & \text{for } k = 1,2\,, \\ +1\,, & \text{for } k = 3\,, \end{cases}$$

then,

$$\operatorname{Tr}\big[(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}})\tau_k\big] = x_j \operatorname{Tr}\tau_j\tau_k = 2\varepsilon_k x_k\,, \qquad (\text{no sum over } k)\,.$$

It follows that:

$$x_k = \tfrac{1}{2}\varepsilon_k \operatorname{Tr}\big[(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}})\tau_k\big]\,. \tag{12}$$

Note that $\tau_k^\dagger = \varepsilon_k\tau_k$ for $k \in \{1, 2, 3\}$, although we shall not make use of this notation here.

Next, we write:

$$\vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}} = V\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}}\, V^{-1}\,, \qquad \text{where } V \in \mathrm{SU}(1\,,1)\,, \tag{13}$$

from which it follows that

$$\det \vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}} = \det(V\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}}\, V^{-1}) = \det \vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}}\,.$$

Hence,

$$x_1^2 + x_2^2 - x_3^2 = y_1^2 + y_2^2 - y_3^2\,.$$

This is equivalent to the statement that

$$y_i = \mathcal{R}_{ij}x_j\,, \qquad \text{where } \mathcal{R} \in \mathrm{SO}(2\,,1)\,, \tag{14}$$

since $\mathrm{O}(2\,,1)$ transformations preserve the quadratic form $x_1^2+x_2^2-x_3^2$. Moreover, $\det\mathcal{R} = 1$, since $\mathcal{R}$ is continuously connected to the identity. Thus, it follows that $\mathcal{R} \in \mathrm{SO}(2\,,1)$.

Hence, eqs. (12) and (14) yield

$$\mathcal{R}_{jk} = \tfrac{1}{2}\varepsilon_k \mathrm{Tr}(V\tau_k V^{-1}\tau_j)\,, \tag{15}$$

which is a homomorphism of $\mathrm{SU}(1\,,1)$ onto $\mathrm{SO}(2\,,1)$ that maps

$$\{V\,,\,-V\} \longmapsto \mathcal{R}\,, \qquad \text{where } V \in \mathrm{SU}(1\,,1) \text{ and } \mathcal{R} \in \mathrm{SO}(2\,,1)\,.$$

As in part (b), it is easy to check that the multiplication law is preserved (the proof is almost identical to the one previously given and will not be repeated here). From the explicit form for $\mathcal{R}_{jk}$ given in eq. (15), it is clear that both $V$ and $-V$ correspond to the same $\mathrm{SO}(2\,,1)$ element $\mathcal{R}$. Equivalently, the kernel of the homomorphism defined by eq. (15) is $\mathbb{Z}_2 \cong \{\mathbf{I}\,,\,-\mathbf{I}\}$, as these are the only two $\mathrm{SU}(1\,,1)$ matrices that are mapped onto the identity element, $\mathcal{R}_{ij} = \delta_{ij}$ of $\mathrm{SO}(2\,,1)$. Hence, the homomorphism defined by eq. (15) is a two-to-one mapping. Hence, one can introduce an equivalence relation among $\mathrm{SU}(1\,,1)$ matrices such that $V_1 \sim V_2$ if either $V_1 = V_2$ or $V_1 = -V_2$. It therefore follows that

$$\mathrm{SO}(2\,,1) \cong \mathrm{SU}(1\,,1)/\mathbb{Z}_2\,.$$

**Remarks**

Following the remarks at the end of part (b), we can again employ eqs. (4) and (5) to the defining representation of SU(1,1), by identifying $g = V$, $\mathcal{A}_k = \frac{1}{2}\tau_k$ and $D_{ij} = \mathcal{R}_{ij}$, where $\mathcal{R}$, whose explicit form is given by eq. (15), are the matrices of the adjoint representation of SU(1,1), which correspond to the elements of the defining representation of SO(2,1). That is,

$$V\tau_j V^{-1} = \mathcal{R}_{ij}\tau_i\,. \tag{16}$$

Multiplying both sides of eq. (16) by $x_j$ yields,

$$V\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}}\,V^{-1} = x_j\mathcal{R}_{ij}\tau_i = \vec{\boldsymbol{y}}\boldsymbol{\cdot}\vec{\boldsymbol{\tau}}\,, \tag{17}$$

where $y_i = \mathcal{R}_{ij}x_j$, thereby reproducing eq. (13).

2. The Möbius group is defined as the set of linear fractional transformations:

$$M = \left\{ m(z) = \frac{az+b}{cz+d}\,, \quad ad-bc=1 \right\}\,,$$

where $a$, $b$, $c$, $d$ and $z$ are complex numbers.

(a) Show that the mapping $f : \mathrm{SL}(2,\mathbb{C}) \to M$ defined by:

$$f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto m(z) \tag{18}$$

is a group homomorphism.

To demonstrate the homomorphism, we first consider the group multiplication of $\mathrm{SL}(2,\mathbb{C})$,

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix}.$$

Compare this with the product of two linear fractional transformations. If we define

$$m_1(z) = \frac{a_1z+b_1}{c_1z+d_1}\,,$$

then the composition of two successive linear fractional transformations is given by:

$$m_2(m_1(z)) = \frac{a_2\left(\dfrac{a_1z+b_1}{c_1z+d_1}\right)+b_2}{c_2\left(\dfrac{a_1z+b_1}{c_1z+d_1}\right)+d_2} = \frac{a_2(a_1z+b_1)+b_2(c_1z+d_1)}{c_2(a_1z+b_1)+d_2(c_1z+d_1)}$$

$$= \frac{(a_2a_1+b_2c_1)z + a_2b_1 + b_2d_1}{(c_2a_1+d_2c_1)z + c_2b_1 + d_2d_1}\,.$$

Thus, the multiplication law of $\mathrm{SL}(2,\mathbb{C})$ is preserved. Moreover, the identity of $\mathrm{SL}(2,\mathbb{C})$, denoted by $\mathbf{I}$ (the $2\times 2$ identity matrix), corresponding to $a=d=1$ and $b=c=0$ is mapped to the identity linear fractional transformation $m(z)=z$. Finally, one can easily verify that the inverse $\mathrm{SL}(2,\mathbb{C})$ matrix,[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{19}$$

is mapped to the inverse linear fractional transformation,

$$m^{-1}(z) = \frac{dz-b}{-cz+a}\,,$$

[1]In computing the inverse of the matrix in eq. (19), we have employed the relation $ad-bc=1$.

since $mm^{-1}(z) = z$ as shown by the following computation (which uses $ad - bc = 1$),

$$m(m^{-1}(z)) = \frac{a\left(\dfrac{dz-b}{-cz+a}\right)+b}{c\left(\dfrac{dz-b}{-cz+a}\right)+d} = \frac{a(dz-b)+b(-cz+a)}{c(dz-b)+d(-cz+a)} = z\,.$$

Hence, the mapping specified by eq. (18) is a homomorphism. It is not an isomorphism since for any $T \in \mathrm{SL}(2,\mathbb{C})$, the two elements $T$ and $-T$ are mapped into the same element of $M$. The kernel of the mapping $f$ is $\mathbb{Z}_2 = \{\mathbf{I}\,,\,-\mathbf{I}\}$. We conclude that $M \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

(b) Prove that $M$ is not simply connected and identify its universal covering group.

$M$ is *not* simply connected. Consider a path in the $\mathrm{SL}(2,\mathbb{C})$ manifold that starts at the identity of the group manifold, $\mathbf{I} \in \mathrm{SL}(2,\mathbb{C})$, and ends at $-\mathbf{I} \in \mathrm{SL}(2,\mathbb{C})$. If we map this path onto $M$ it corresponds to a closed curve since $\mathbf{I}$ and $-\mathbf{I}$ are both mapped to the identity of $M$. Hence, this is a closed path in $M$ that cannot be contracted to a point. Hence, we conclude that $M$ is not simply connected. In fact, $M$ is doubly connected just like SO(3). The universal covering group of $M$ is $\mathrm{SL}(2,\mathbb{C})$, which is a simply connected group.

To prove that $\mathrm{SL}(2,\mathbb{C})$ is simply connected, we shall employ the polar decomposition of an invertible complex matrix $S$. Namely, $S$ can be uniquely written as $S = HU$ where $H$ is a positive definite hermitian matrix and $U$ is unitary.[2] Note that $|\det U| = 1$ and $\det H > 0$, since the eigenvalues of $H$ are all positive. Hence, it follows that any matrix $T \in \mathrm{SL}(2,\mathbb{C})$, which by definition has unit determinant, can be uniquely expressed as the product $T = PU$, where $U \in \mathrm{SU}(2)$ and $P$ is a $2\times 2$ positive definite hermitian matrix with unit determinant. Moreover, one can uniquely write $P = e^X = \exp\{\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\}$ where $\vec{\boldsymbol{x}} \in \mathbb{R}^3$ and $X$ is a $2\times 2$ traceless hermitian matrix.[3] That is, the set of all $2\times 2$ positive definite hermitian matrices with unit determinant is topologically equivalent to $\mathbb{R}^3$.

Thus, there is a continuous invertible map, $T \mapsto (U\,,\,\vec{\boldsymbol{x}})$, from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SU}(2)\times\mathbb{R}^3$. Using the fact that $\mathrm{SU}(2) \simeq S^3$ (as shown in class), it follows that $\mathrm{SL}(2,\mathbb{C})$ is topologically equivalent to the direct product space $S^3\times\mathbb{R}^3$. Since both $S^3$ and $\mathbb{R}^3$ are simply connected, so is their direct product. Hence, $\mathrm{SL}(2,\mathbb{C})$ is simply connected.[4]

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[2]To prove this result, we first note that $SS^\dagger$ is a positive definite hermitian matrix. Thus, we define $H \equiv (SS^\dagger)^{1/2}$, which is the unique positive definite hermitian square root of $SS^\dagger$. One can now show that $U$ is unitary since $U = H^{-1}S$, in which case $U^\dagger U = S^\dagger H^{-1}H^{-1}S = S^\dagger(SS^\dagger)^{-1}S = \mathbf{I}$. For further details (including a proof that the polar decomposition is unique), see e.g. Chapter 10 of Denis Serre, *Matrices: Theory and Applications*, Second Edition (Springer Science, New York, 2010).

[3]The most general $2\times 2$ traceless hermitian matrix can be written as $\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}$ for some real three-vector, $\vec{\boldsymbol{x}} \in \mathbb{R}^3$. One can easily compute $\exp\{\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\} = \cosh x + (\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})\sinh x$, where $x \equiv |\vec{\boldsymbol{x}}|$, and show that this matrix is a positive definite hermitian matrix with unit determinant. In particular, the eigenvalues of $\exp\{\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\} = \exp\{x\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\}$ are easily seen to be $e^x$ and $e^{-x}$ (since the eigenvalues of $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}$ are $\pm 1$), which are manifestly positive. Moreover, allowing $\vec{\boldsymbol{x}}$ to range over all possible vectors in $\mathbb{R}^3$ yields all possible positive definite hermitian matrices with unit determinant, $\exp\{\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\}$. Hence there exists a map from $P$ to $\vec{\boldsymbol{x}}$ that is one-to-one and onto.

[4]For further details, see e.g. Brian C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, 2nd edition (Springer International Publishing, Cham, Switzerland, 2015).

3. In class, we showed that the invariant measure on a Lie group manifold is given by

$$d\mu(g) = |\det c(\vec{\boldsymbol{\xi}})|\, d\xi_1 d\xi_2 \cdots d\xi_n\,, \tag{20}$$

where the matrix elements $c_{jk}(\vec{\boldsymbol{\xi}})$ are the coefficients of the Lie algebra element $A^{-1}\partial A/\partial\xi_k$ with respect to some basis, and $A(\vec{\boldsymbol{\xi}})$ are elements of the corresponding Lie group that is parametrized by the coordinates $\vec{\boldsymbol{\xi}}$. That is, given an $n$-dimensional Lie group $G$, the corresponding real Lie algebra $\mathfrak{g}$ consists of real linear combinations of basis vectors $\mathcal{A}_j \in \mathfrak{g}$. Since $A^{-1}\partial A/\partial\xi_k \in \mathfrak{g}$ for any $A \in G$, one can therefore write,[5]

$$A^{-1}\frac{\partial A}{\partial \xi_k} = \sum_{j=1}^{n} c_{jk}(\vec{\boldsymbol{\xi}})\mathcal{A}_j\,, \tag{21}$$

which defines the coefficients $c_{jk}(\vec{\boldsymbol{\xi}})$ needed in the determination of the invariant measure.

(a) An element of SO(3) can be parametrized by $\vec{\boldsymbol{\xi}} = (\alpha,\beta,\gamma)$, where $\alpha$, $\beta$ and $\gamma$ are the three Euler angles defined in Appendix E of the class handout entitled *Properties of Proper and Improper Rotation Matrices*. Using the Euler angle parametrization of the SO(3) group manifold, compute the invariant integration measure $d\mu(g)$ for SO(3).

In the class handout cited above, an explicit expression for $R(\alpha,\beta,\gamma) \in \mathrm{SO}(3)$ is given by,

$$R(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}, \tag{22}$$

where $0 \le \alpha < 2\pi$, $0 \le \beta \le \pi$ and $0 \le \gamma < 2\pi$.

In order to employ eq. (21), we evaluate the following derivatives

$$\frac{\partial R}{\partial \alpha} = \begin{pmatrix} -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\cos\beta\sin\gamma - \cos\alpha\cos\gamma & -\sin\alpha\sin\beta \\ \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

$$\frac{\partial R}{\partial \beta} = \begin{pmatrix} -\cos\alpha\sin\beta\cos\gamma & \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \\ -\sin\alpha\sin\beta\cos\gamma & \sin\alpha\sin\beta\sin\gamma & \sin\alpha\cos\beta \\ -\cos\beta\cos\gamma & \cos\beta\sin\gamma & -\sin\beta \end{pmatrix}, \tag{24}$$

$$\frac{\partial R}{\partial \gamma} = \begin{pmatrix} -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & -\cos\alpha\cos\beta\cos\gamma + \sin\alpha\sin\gamma & 0 \\ -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & 0 \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & 0 \end{pmatrix}. \tag{25}$$

Matrix multiplication then yields,

$$R^{-1}\frac{\partial R}{\partial \alpha} = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\gamma \\ \cos\beta & 0 & \sin\beta\cos\gamma \\ -\sin\beta\sin\gamma & -\sin\beta\cos\gamma & 0 \end{pmatrix}, \tag{26}$$

[5] Consider the analytic curve in the group manifold, $C_k(t) = A^{-1}(\xi_1,\ldots,\xi_n)A(\xi_1,\ldots,\xi_k+t,\ldots,\xi_n)$. Note that $C_k(0) = \boldsymbol{I}$ so that the curve passes through the identity at $t=0$. As shown in class, $(dC_k/dt)_{t=0} \in \mathfrak{g}$. But a simple calculation yields $(dC_k/dt)_{t=0} = A^{-1}\partial A/\partial\xi_k$. Hence, it follows that $A^{-1}\partial A/\partial\xi_k \in \mathfrak{g}$.

$$R^{-1}\frac{\partial R}{\partial\beta} = \begin{pmatrix} 0 & 0 & \cos\gamma \\ 0 & 0 & -\sin\gamma \\ -\cos\gamma & \sin\gamma & 0 \end{pmatrix}\,, \tag{27}$$

$$R^{-1}\frac{\partial R}{\partial\gamma} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \tag{28}$$

where we have used $R^{-1} = R^{\mathsf{T}}$ (since $R$ is an orthogonal matrix). As expected, $R^{-1}\partial R/\partial\xi_k$ [where $\vec{\boldsymbol{\xi}} = (\alpha,\beta,\gamma)$] are real antisymmetric $3\times 3$ matrices and hence elements of the $\mathfrak{so}(3)$ Lie algebra. The standard basis for $\mathfrak{so}(3)$ is given by $(\mathcal{A}_i)_{jk} = -\epsilon_{ijk}$. That is,

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\,, \quad \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{29}$$

Using eqs. (26)–(28), we can read off the matrix $C$ whose coefficients, $c_{jk}(\vec{\boldsymbol{\xi}})$, are defined in eq. (21),

$$C = \begin{pmatrix} -\sin\beta\cos\gamma & \sin\gamma & 0 \\ \sin\beta\sin\gamma & \cos\gamma & 0 \\ \cos\beta & 0 & 1 \end{pmatrix}\,. \tag{30}$$

Since $\det C = -\sin\beta$ (and $0 \leq \beta \leq \pi$ so that $|\det C| = \sin\beta$), we conclude that the invariant measure of SO(3) is given by,

$$d\mu(\alpha,\beta,\gamma) = \sin\beta\, d\alpha\, d\beta\, d\gamma\,. \tag{31}$$

*REMARKS*:

Note that $d\mu(\alpha,\beta,\gamma)$ vanishes at $\beta = 0$ and at $\beta = \pi$. This is related to the observation that

$$R(\alpha, \sin\beta = 0, \gamma) = \begin{pmatrix} \pm\cos(\gamma\pm\alpha) & \mp\sin(\gamma\pm\alpha) & 0 \\ \sin(\gamma\pm\alpha) & \cos(\gamma\pm\alpha) & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}\,, \tag{32}$$

where upper signs correspond to $\beta = 0$ and lower signs correspond to $\beta = \pi$, which implies that $R(\alpha, \sin\beta = 0, \gamma)$ is independent of $\gamma \mp \alpha$. That is, the Euler angle parametrization is not unique for $\beta = 0$ and for $\beta = \pi$. In particular, in light of $R(\alpha, 0, 2\pi - \alpha) = \mathbf{I}$, independently of the value of $\alpha$, it follows that the Euler angle parametrization is singular at the identity element of SO(3).[6] One consequence of this observation is that one cannot use this parametrization to obtain the basis for the $\mathfrak{so}(3)$ Lie algebra by using,

$$\mathcal{A}_k = \left.\frac{\partial R}{\partial\xi_k}\right|_e\,, \tag{33}$$

since this formula assumes that the coordinates that specifies the group elements is nonsingular at the identity $e$.

[6] This point has been emphasized in Chapter 9, sections 1 and 2, of James D. Talman, *Special Functions: A Group Theoretic Approach* (W.A. Benjamin, Inc., New York, NY, 1968).

Note that if eq. (33) were valid, then one could set $\vec{\boldsymbol{\xi}} = 0$ (corresponding to the identity element) in eq. (21). Using eq. (33) along with the fact that $A^{-1} = \mathbf{I}$ at the identity element would then yield,

$$c_{jk}(\vec{\mathbf{0}}) = \delta_{jk}\,. \tag{34}$$

In the case of $\mathfrak{so}(3)$, it would then follow that $C(\vec{\mathbf{0}}) = \mathbf{I}$, which is violated by eq. (30). This violation is consistent with the observation that eq. (33) cannot be used in this case because the Euler angle parametrization is singular at the identity element of SO(3).

Although the singular behavior of the Euler angle parametrization of SO(3) at the identity is annoying, there is a simple fix. Namely, in the neighborhood of the identity element, one can employ the angle-and-axis parametrization of the SO(3) group manifold (whose behavior is nonsingular at the identity). Outside the neighborhood of the identity (and more generally, excluding regions where $\beta = 0$ and $\beta = \pi$), the Euler angle parametrization of the SO(3) group manifold is nonsingular. In the overlap region in which both coordinate systems are valid, the Euler angles and the angle-and-axis parameters are smoothly related. Hence, it is valid to employ the standard basis for the $\mathfrak{so}(3)$ Lie algebra in eq. (21) to determine the matrix $C$ as we did in eqs. (29) and (30). Thus, the result obtained in eq. (31) is correct. The vanishing of $d\mu$ at $\beta = 0$ and $\pi$ is simply an indication of the non-uniqueness of the Euler angle parametrization of the SO(3) group manifold as indicated below eq. (32).

It should be noted that the same phenomenon noted above is also at play when considering the measure for three-dimensional Euclidean space ($\mathbb{R}^3$) using spherical coordinates $(r, \theta, \phi)$. It is well known that this coordinate system is singular at $\theta = 0$ and $\theta = \pi$, since at these values, the coordinate $\phi$ is undefined. In addition, this coordinate system is singular at $r = 0$ since in this case, both $\theta$ and $\phi$ are undefined. This is reflected by the fact that the measure in spherical coordinates, $d^3\boldsymbol{x} = r^2 \sin\theta\, dr\, d\theta\, d\phi$, vanishes for $\sin\theta = 0$ and for $r = 0$.

(b) The SO(3) group manifold can be also be described as a ball of radius $\pi$ with antipodal points identified. A point in the SO(3) group manifold is specified by a vector $\vec{\boldsymbol{\xi}}$ with $|\vec{\boldsymbol{\xi}}| \leq \pi$. Thus, the SO(3) manifold is parametrized by $\vec{\boldsymbol{\xi}} = (\xi, \theta, \phi)$, where $(\theta, \phi)$ are the spherical angles (such that $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$) and $\xi$ is the magnitude of the vector $\vec{\boldsymbol{\xi}}$. [*NOTE:* This is equivalent to the angle-and-axis parametrization where the rotation angle is $\xi$ and the rotation axis, $\hat{\boldsymbol{\xi}}$, is specified by a polar angle $\theta$ and an azimuthal angle $\phi$.]

Show that the the matrix elements of $c(\vec{\boldsymbol{\xi}})$ defined in eq. (21) are given by,

$$c(\vec{\boldsymbol{\xi}})_{nk} = \tfrac{1}{2}\epsilon_{\ell nj} R^{-1}_{\ell i} \frac{dR_{ij}}{d\xi_k}\,, \tag{35}$$

and $R_{ij} \equiv R_{ij}(\vec{\boldsymbol{\xi}})$ is the SO(3) matrix given in problem 7(b) of problem set 2.

The most general element of the Lie group SO(3) is given in problem 7(b) of problem set 2 [cf. eq. (2)],

$$R_{ij}(\vec{\boldsymbol{\xi}}) = \delta_{ij}\cos\xi + \xi_i\xi_j\left(\frac{1-\cos\xi}{\xi^2}\right) - \epsilon_{ijm}\xi_m \frac{\sin\xi}{\xi}\,, \tag{36}$$

where $\vec{\boldsymbol{\xi}} \equiv \xi\hat{\boldsymbol{n}}$ and $\xi \equiv |\vec{\boldsymbol{\xi}}|$. The matrix generators of the SO(3) Lie algebra are defined by:

$$\mathcal{A}_m = \left.\frac{\partial R}{\partial \xi_m}\right|_{\vec{\boldsymbol{\xi}}=0} .$$

It suffices to consider infinitesimal $\xi$, in which case,

$$R_{ij}(\vec{\boldsymbol{\xi}}) = \delta_{ij} - \epsilon_{ijm}\xi_m + \mathcal{O}(\xi^2) .$$

Hence,

$$(\mathcal{A}_m)_{ij} = -\epsilon_{ijm} .$$

Using the above results, eq. (21) takes the following form:

$$R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} = -\epsilon_{\ell jm}c_{mk}(\vec{\boldsymbol{x}}) .$$

Multiplying both sides of this equation by $\epsilon_{\ell nj}$ and using the identity, $\epsilon_{\ell nj}\epsilon_{\ell jm} = -2\delta_{mn}$, one obtains:

$$c(\vec{\boldsymbol{\xi}})_{nk} = \tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} . \tag{37}$$

(c) Using eqs. (20) and (35), evaluate the invariant integration measure $d\mu(g)$ for the angle-and-axis parametrization of SO(3) and show that

$$d\mu(\vec{\boldsymbol{\xi}}) = 2(1-\cos\xi)\sin\theta\, d\theta\, d\phi\, d\xi .$$

We now compute $\det c(\vec{\boldsymbol{\xi}})$ given in eq. (37). Recall that $[R(\hat{\boldsymbol{n}},\xi)]^{-1} = R(-\hat{\boldsymbol{n}},\xi)$. Using $\vec{\boldsymbol{\xi}} = \xi\hat{\boldsymbol{n}}$, it follows that $R^{-1}(\vec{\boldsymbol{\xi}}) = R(-\vec{\boldsymbol{\xi}})$. Hence,

$$R^{-1}_{ij}(\vec{\boldsymbol{\xi}}) = \delta_{ij}\cos\xi + \xi_i\xi_j\left(\frac{1-\cos\xi}{\xi^2}\right) + \epsilon_{ijm}\xi_m\frac{\sin\xi}{\xi} . \tag{38}$$

It follows that:

$$\epsilon_{\ell nj}R^{-1}_{\ell i} = \epsilon_{inj}\cos\xi + \xi_\ell\xi_i\epsilon_{\ell nj}\left(\frac{1-\cos\xi}{\xi^2}\right) + (\delta_{in}\xi_j - \delta_{ij}\xi_n)\frac{\sin\xi}{\xi} , \tag{39}$$

after employing the identity, $\epsilon_{\ell nj}\epsilon_{\ell im} = \delta_{in}\delta_{jm} - \delta_{ij}\delta_{nm}$ and summing over the repeated index $m$. Next, we make use of the chain rule,

$$\frac{dR_{ij}}{d\xi_k} = \frac{\partial R_{ij}}{\partial\xi_k} + \frac{\partial R_{ij}}{\partial\xi}\frac{\partial\xi}{\partial\xi_k} = \frac{\partial R_{ij}}{\partial\xi_k} + \frac{\xi_k}{\xi}\frac{\partial R_{ij}}{\partial\xi} ,$$

where in the last step, we used $\xi \equiv (\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}$ to compute

$$\frac{\partial\xi}{\partial\xi_k} = \frac{\xi_k}{(\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}} = \frac{\xi_k}{\xi} .$$

Hence, using eq. (36),

$$\frac{dR_{ij}}{d\xi_k} = \big(\xi_i\delta_{jk} + \xi_j\delta_{ik}\big)\left(\frac{1-\cos\xi}{\xi^2}\right) - \epsilon_{ijk}\frac{\sin\xi}{\xi} + \frac{\xi_k}{\xi}\left[-\delta_{ij}\sin\xi + \xi_i\xi_j\left(\frac{\sin\xi}{\xi^2} - \frac{2}{\xi^2}(1-\cos\xi)\right) - \epsilon_{ijm}\xi_m\left(\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right)\right]. \quad (40)$$

Multiplying eqs. (39) and (40) and employing the relevant identities for the product of Levi-Civita tensors, we obtain:

$$\begin{aligned}
\tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} &= \tfrac{1}{2}(\epsilon_{ink}\xi_i + \epsilon_{knj}\xi_j)\cos\xi\left(\frac{1-\cos\xi}{\xi^2}\right) + \delta_{n\ell}\frac{\cos\xi\sin\xi}{\xi} + \xi_n\xi_k\frac{\cos\xi}{\xi}\left(\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right) \\
&\quad + \tfrac{1}{2}\epsilon_{\ell nk}\xi_\ell(1-\cos\xi)\left(\frac{1-\cos\xi}{\xi^2}\right) + \tfrac{1}{2}(\delta_{i\ell}\delta_{kn} - \delta_{in}\delta_{k\ell})\xi_i\xi_\ell\frac{\sin\xi}{\xi}\left(\frac{1-\cos\xi}{\xi^2}\right) \\
&\quad + \tfrac{1}{2}(\xi_i\delta_{jk} + \xi_j\delta_{ik})(\xi_j\delta_{in} - \xi_n\delta_{ij})\frac{\sin\xi}{\xi}\left(\frac{1-\cos\xi}{\xi^2}\right) - \tfrac{1}{2}\epsilon_{njk}\xi_j\left(\frac{\sin^2\xi}{\xi^2}\right) \\
&\quad - \tfrac{1}{2}\delta_{ij}(\xi_j\delta_{in} - \xi_n\delta_{ij})\xi_k\left(\frac{\sin^2\xi}{\xi^2}\right) + \tfrac{1}{2}\xi_i\xi_j\xi_k(\xi_j\delta_{in} - \xi_n\delta_{ij})\frac{\sin\xi}{\xi^2}\left[\frac{\sin\xi}{\xi^2} - \frac{2}{\xi^2}(1-\cos\xi)\right].
\end{aligned}$$

The first and last terms above vanish. Collecting the remaining terms, we find:

$$\begin{aligned}
\tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} &= \delta_{nk}\left[\frac{\cos\xi\sin\xi}{\xi} + \frac{\sin\xi}{\xi}(1-\cos\xi)\right] + \tfrac{1}{2}\epsilon_{nk\ell}\xi_\ell\left[\frac{(1-\cos\xi)^2}{\xi^2} + \frac{\sin^2\xi}{\xi^2}\right] \\
&\quad + \xi_n\xi_k\left[\frac{\cos\xi}{\xi}\left(\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right) - \frac{\sin\xi}{\xi}\left(\frac{1-\cos\xi}{\xi^2}\right) + \frac{\sin^2\xi}{\xi^2}\right].
\end{aligned}$$

Simplifying the trigonometric factors, we end up with

$$c(\vec{\boldsymbol{\xi}})_{nk} = \tfrac{1}{2}\epsilon_{\ell nj}R^{-1}_{\ell i}\frac{dR_{ij}}{d\xi_k} = \left(\delta_{nk} - \frac{\xi_n\xi_k}{\xi^2}\right)\frac{\sin\xi}{\xi} + \frac{\xi_n\xi_k}{\xi^2} + \epsilon_{nk\ell}\xi_\ell\left(\frac{1-\cos\xi}{\xi^2}\right). \quad (41)$$

Note that eq. (34) is satisfied as expected, since the angle-and-axis parametrization of the SO(3) group manifold is nonsingular at the identity.

The invariant measure is given by eq. (20). To evaluate $\det c(\vec{\boldsymbol{\xi}})$, we can use the following trick. Consider the matrix $QcQ^{\mathsf{T}}$, where $Q$ is an orthogonal matrix of unit determinant that is chosen such that $Q_{ij}\xi_j = \hat{z}_i$, where $\hat{\boldsymbol{z}}$ is a unit vector pointing in the $z$-direction. Then,

$$(Qc\,Q^{\mathsf{T}})_{mp} = Q_{mc}c_{nk}Q_{pk} = \left(\delta_{mp} - \frac{\hat{z}_m\hat{z}_p}{\xi^2}\right)\frac{\sin\xi}{\xi} + \frac{\hat{z}_m\hat{z}_p}{\xi^2} + Q_{mn}Q_{pk}\epsilon_{nk\ell}\xi_\ell\left(\frac{1-\cos\xi}{\xi^2}\right). \quad (42)$$

Since $Q$ is an orthogonal matrix of unit determinant, it follows that

$$Q_{mn}Q_{pk}\epsilon_{nk\ell} = Q_{r\ell}\epsilon_{mpr}\det Q = Q_{r\ell}\epsilon_{mpr}. \quad (43)$$

Hence, it follows that

$$(Qc\,Q^{\mathsf{T}})_{mp} = \left(\delta_{mp} - \frac{\hat{z}_m\hat{z}_p}{\xi^2}\right)\frac{\sin\xi}{\xi} + \frac{\hat{z}_m\hat{z}_p}{\xi^2} + \epsilon_{mpr}\hat{z}_r\left(\frac{1-\cos\xi}{\xi^2}\right). \quad (44)$$

That is,

$$
Q c\, Q^{\mathsf{T}} = \begin{pmatrix} \dfrac{\sin\xi}{\xi} & \dfrac{1-\cos\xi}{\xi} & 0 \\ -\left(\dfrac{1-\cos\xi}{\xi}\right) & \dfrac{\sin\xi}{\xi} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{45}
$$

Since $\det c = \det(Qc\,Q^{\mathsf{T}})$, the determinant of this matrix is now easily computed,

$$
\det c(\vec{\boldsymbol{\xi}}) = \det(Qc\,Q^{\mathsf{T}}) = \frac{1}{\xi^2}\left[\sin^2\xi + (1-\cos\xi)^2\right] = \frac{2(1-\cos\xi)}{\xi^2}\,, \tag{46}
$$

which is a nonnegative quantity. Consequently, the invariant measure of SO(3) is given by:[7]

$$
d\mu(\vec{\boldsymbol{\xi}}) = \frac{2(1-\cos\xi)}{\xi^2}\, d^3\xi = 2(1-\cos\xi)\sin\theta\, d\theta\, d\phi\, d\xi\,. \tag{47}
$$

For completeness, I have provided in Appendix A yet another trick for directly computing $\det c(\vec{\boldsymbol{\xi}})$ without rotating the vector $\vec{\boldsymbol{\xi}}$ to point in the $z$-direction.

*REMARKS*:

It is instructive to evaluate $|\det c(\vec{\boldsymbol{\xi}})|$ by first computing the matrix $c^{\mathsf{T}}c$.

$$
\begin{aligned}
(c^{\mathsf{T}}c)_{mk} = c_{nm}c_{nk} = &\left[\left(\delta_{nm} - \frac{\xi_n\xi_m}{\xi^2}\right)\frac{\sin\xi}{\xi} + \frac{\xi_n\xi_m}{\xi^2} + \epsilon_{nm\ell}\xi_\ell\left(\frac{1-\cos\xi}{\xi^2}\right)\right] \\
&\times\left[\left(\delta_{nk} - \frac{\xi_n\xi_k}{\xi^2}\right)\frac{\sin\xi}{\xi} + \frac{\xi_n\xi_k}{\xi^2} + \epsilon_{nk\ell}\xi_\ell\left(\frac{1-\cos\xi}{\xi^2}\right)\right],
\end{aligned} \tag{48}
$$

with an implicit sum over the repeated index $n$. We can evaluate eq. (48) by using the following identities:

$$
\left(\delta_{nm} - \frac{\xi_n\xi_m}{\xi^2}\right)\left(\delta_{nk} - \frac{\xi_n\xi_k}{\xi^2}\right) = \left(\delta_{mk} - \frac{\xi_m\xi_k}{\xi^2}\right)\,, \tag{49}
$$

$$
\left(\delta_{nm} - \frac{\xi_n\xi_m}{\xi^2}\right)\frac{\xi_n\xi_k}{\xi^2} = 0\,, \tag{50}
$$

$$
\left(\delta_{nm} - \frac{\xi_n\xi_m}{\xi^2}\right)\epsilon_{nk\ell}\xi_\ell = \epsilon_{mk\ell}\xi_\ell\,, \tag{51}
$$

$$
\epsilon_{nm\ell}\epsilon_{nkj}\frac{\xi_\ell\xi_j}{\xi^2} = (\delta_{mk}\delta_{\ell j} - \delta_{mj}\delta_{k\ell})\frac{\xi_\ell\xi_j}{\xi^2} = \delta_{mk} - \frac{\xi_m\xi_k}{\xi^2}\,. \tag{52}
$$

It then follows that

$$
(c^{\mathsf{T}}c)_{mk} = \left(\delta_{mk} - \frac{\xi_m\xi_k}{\xi^2}\right)\left[\frac{\sin^2\xi + (1-\cos\xi)^2}{\xi^2}\right] + \frac{\xi_m\xi_k}{\xi^2} + \frac{\sin\xi(1-\cos\xi)}{\xi^3}(\epsilon_{mk\ell} + \epsilon_{m\ell k})\xi_\ell\,. \tag{53}
$$

[7] In spherical coordinates, $\vec{\boldsymbol{\xi}} = \xi(\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)$ and $d^3\xi = \xi^2 d\xi d\Omega = \xi^2 d\xi\sin\theta d\theta d\phi$.

The last term of eq. (53) vanishes due to the antisymmetry of the Levi-Civita tensor. After further simplification, we arrive at[8]

$$(c^{\mathsf{T}}c)_{mk} = \frac{2(1-\cos\xi)}{\xi^2}\left(\delta_{mk} - \frac{\xi_m\xi_k}{\xi^2}\right) + \frac{\xi_m\xi_k}{\xi^2}\,. \tag{54}$$

To compute the determinant of $c^{\mathsf{T}}c$, we simply use the definition

$$\det\, A = \epsilon_{ijk}A_{1i}A_{2j}A_{3k}\,,$$

where there is an implicit threefold sum over $i$, $j$, $k$. Given a $3\times 3$ matrix of the form,

$$A_{ij} = b^2\delta_{ij} + a_i a_j\,,$$

it follows that

$$\begin{aligned}\det\, A &= \epsilon_{ijk}(b^2\delta_{1i} + a_1a_i)(b^2\delta_{2j} + a_2a_j)(b^2\delta_{1k} + a_3a_k) \\ &= b^6 + b^4\big[a_1a_{i1}\epsilon_{i23} + a_2a_j\epsilon_{1j3} + a_3a_k\epsilon_{12k}\big]\,,\end{aligned} \tag{55}$$

since $\epsilon_{123} = 1$, and the terms proportional to $b^2$ and the terms independent of $b$ vanish due to the antisymmetry of $\epsilon_{ijk}$ (e.g., $\epsilon_{ijk}a_ja_k = 0$, etc.). Hence, eq. (55) yields

$$\det\, A = b^4[b^2 + a_1^2 + a_2^2 + a_3^2]\,. \tag{56}$$

We can now identify:

$$b^2 = \frac{2(1-\cos\xi)}{\xi^2}\,, \qquad a_i = \frac{\xi_i(1-b^2)^{1/2}}{\xi}\,. \tag{57}$$

It follows that $b^2 + a_1^2 + a_2^2 + a_3^2 = 1$. Therefore, eqs. (54), (56), and (57) yield

$$\det(c^{\mathsf{T}}c) = \frac{4(1-\cos\xi)^2}{\xi^4}\,. \tag{58}$$

The quantity $g_{mk} \equiv (c^{\mathsf{T}}c)_{mk}$ corresponds to the invariant metric on the SO(3) manifold. Denoting $g \equiv \det g$, the invariant measure can be identified as $d\mu(\vec{\boldsymbol{\xi}}) = \sqrt{|g|}\, d^3\xi$, in agreement with eq. (46).

*Relating the results of parts (a) and (c)*:

To see the consistency of the results obtained in parts (a) and (c), it is instructive to rederive eq. (47) from eq. (31) by computing the Jacobian of the transformation from Euler angles to angle-and-axis parameters. Here, we can employ the results of Appendix E of the class handout entitled *Properties of Proper and Improper Rotation Matrices.* It is convenient to first define

$$x \equiv \tfrac{1}{2}(\alpha+\gamma)\,, \qquad y \equiv \tfrac{1}{2}(\alpha-\gamma)\,. \tag{59}$$

[8]It is noteworthy that at the origin of the SO(3) manifold corresponding to $\vec{\boldsymbol{\xi}} = \vec{\mathbf{0}}$, we find the expected behavior, $(c^{\mathsf{T}}c)_{mk} = \delta_{mk}$.

Eq. (103) of the handout cited above states that $\phi = \frac{1}{2}\epsilon\pi + y$, which implies that[9]

$$\frac{\partial y}{\partial \xi} = 0\,, \qquad \frac{\partial y}{\partial \theta} = 0\,, \qquad \frac{\partial y}{\partial \phi} = 1\,. \tag{60}$$

Next, eq. (105) of the handout cited above in the notation employed here states that

$$\sin(\beta/2) = \sin\theta \sin(\xi/2)\,. \tag{61}$$

Hence, it follows that

$$\frac{\partial \beta}{\partial \xi} = \frac{\cos(\xi/2)\sin\theta}{\sqrt{1-\sin^2(\xi/2)\sin^2\theta}}\,, \qquad \frac{\partial \beta}{\partial \theta} = \frac{2\sin(\xi/2)\cos\theta}{\sqrt{1-\sin^2(\xi/2)\sin^2\theta}}\,, \qquad \frac{\partial \beta}{\partial \phi} = 0\,. \tag{62}$$

Finally, eqs. (105)–(107) of the handout cited above in the notation employed here gives,

$$\sin x = \frac{\epsilon\sin(\xi/2)\cos\theta}{\sqrt{1-\sin^2(\xi/2)\sin^2\theta}}\,, \qquad \cos x = \frac{\epsilon\cos(\xi/2)}{\sqrt{1-\sin^2(\xi/2)\sin^2\theta}}\,. \tag{63}$$

It follows that

$$\frac{\partial x}{\partial \xi} = \frac{\cos\theta}{2[1-\sin^2(\xi/2)\sin^2\theta]}\,, \qquad \frac{\partial x}{\partial \theta} = -\frac{\sin(\xi/2)\cos(\xi/2)\sin\theta}{1-\sin^2(\xi/2)\sin^2\theta}\,, \qquad \frac{\partial x}{\partial \phi} = 0\,. \tag{64}$$

Hence the Jacobian matrix is

$$J \equiv \frac{\partial(x,\beta,y)}{\partial(\xi,\theta,\phi)} = \begin{pmatrix} \dfrac{\cos\theta}{2[1-\sin^2(\xi/2)\sin^2\theta]} & -\dfrac{\sin(\xi/2)\cos(\xi/2)\sin\theta}{1-\sin^2(\xi/2)\sin^2\theta} & 0 \\ \dfrac{\cos(\xi/2)\sin\theta}{\sqrt{1-\sin^2(\xi/2)\sin^2\theta}} & \dfrac{2\sin(\xi/2)\cos\theta}{\sqrt{1-\sin^2(\xi/2)\sin^2\theta}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{65}$$

The determinant of $J$ is given (after some algebraic manipulations) by

$$\det J = \frac{\sin(\xi/2)}{\sqrt{1-\sin^2(\xi/2)\sin^2\theta}} = \frac{\sin(\xi/2)}{\cos(\beta/2)}\,, \tag{66}$$

where we have used eq. (61) in the final step above.

In light of eq. (59), $d\alpha\, d\gamma = 2\, dx\, dy$. Hence, after employing eqs. (61) and (66),

$$\sin\beta\, d\alpha\, d\beta\, d\gamma = 4\sin(\beta/2)\cos(\beta/2)|\det J| d\xi\, d\theta\, d\phi = 4\sin\theta\, \sin^2(\xi/2)\, d\xi\, d\theta\, d\phi\,, \tag{67}$$

which is equivalent to the result of eq. (47), after using the identity $\sin^2(\xi/2) = \frac{1}{2}(1-\cos\xi)$.

[9]The sign factor, $\epsilon \equiv \text{sgn}\{\cos x \cos(\beta/2)\}$, can be taken to be a constant except in the region where the argument of the sign function vanishes. As this is a region of measure zero, we can ignore it in the computation of the Jacobian.

4. Consider a Lie group of transformations $G$ acting on a manifold $M$. That is, for every $g \in G$, we have $gx = y$ for some $x, y \in M$.

(a) Let $H$ be the set of all transformations in $G$ that map a given point $x \in M$ into itself. Show that $H$ is a subgroup. $H$ has at least three names in the mathematical literature: the little group, the isotropy group, or the stability group of the point $x$.

Suppose that $h \in H$ such that $hx = x$. First, we note that $ex = x$, so that $e \in H$. If $h_1\,,\, h_2 \in H$, then $h_1 h_2 \in H$, since $h_1 h_2 x = h_1 x = x$. Moreover, $hx = x$ implies that $x = h^{-1}x$ (by multiplication by $h^{-1}$ on both sides of the equation), so that $h^{-1} \in H$. Hence, $H$ is a group. Since $H$ is a subset of $G$, we conclude that $H$ is a subgroup of $G$.

(b) Consider the submanifold of $M$ defined by $\{gx \,|\, g \in G\}$, for fixed $x \in M$. This is called the *orbit* through $x$ with respect to $G$. Show that there is a one-to-one correspondence between the points of the orbit and the set of left cosets of $H$. Explain why we may conclude that $\{gx \,|\, g \in G\} = G/H$. Show that the coset space $G/H$ is homogeneous.

The subgroup $H$ is defined as the set of all $h \in H$ such that $hx = x$. This condition can be rewritten as $Hx = x$. To show that there is a one-to-one correspondence between the points in the orbit through $x$ and the left cosets of $H$, consider the following result:

$$g_i H x = g_i x \,, \qquad \text{where } g_i \in G\,.$$

As $g_i$ varies over the group, $g_i H$ ranges over all left cosets of $H$ and $g_i x$ ranges over all points in the orbit through $x$. Thus, all be have to show is

$$g_1 H = g_2 H \quad \Longleftrightarrow \quad g_1 x = g_2 x\,,$$

Note that $g_1 H = g_2 H \implies g_1 = g_2 h$ for some $h \in H$. Hence, $g_1 x = g_2 h x = g_2 x$. Likewise, $g_1 x = g_2 x \implies g_2^{-1} g_1 x = x \implies g_2^{-1} g_1 \in H$. Since $HH = H$ (this is simply the statement that the subgroup $H$ is closed under group multiplication), it follows that $g_2^{-1} g_1 H \in H$. Since two cosets that are not disjoint must coincide, $g_2^{-1} g_1 H = eH = H$. Multiplying both sides by $g_2$ then yields $g_1 H = g_2 H$.

We conclude that the points of the orbit are in one-to-one correspondence with the distinct left cosets of $H$. The set of left cosets is denoted by $G/H$. Hence,

$$G/H = \{gx \,|\, g \in G\}\,, \qquad \text{for a fixed } x \in M\,.$$

To prove that $G/H$ is homogeneous, we must exhibit a homeomorphism $\phi$ that maps any point of $G/H$ to any other point. Given the points $y, z \in G/H$ such that $y = g_1 x$ and $z = g_2 x$, consider the homeomorphism,

$$\phi(gx) = g_2 g_1^{-1} g x\,.$$

For $g = g_1$, we have $\phi(y) = z$. That is, the homogeneous property of $G/H$ derives from the homogeneous property of the Lie group $G$.

(c) Prove that $S^{n-1}$ =SO$(n)$/SO$(n-1)$ by considering the action of the rotation group on the point $(1,0,0,\ldots,0)\in\mathbb{R}^n$.

Consider the action of the rotation group SO$(n)$ on vectors in $\mathbb{R}^n$. Let us focus on the point $\vec{\boldsymbol{x}}\in\mathbb{R}^n$. The orbit through $x$ under the group action SO$(n)$ consists of all vectors $\vec{\boldsymbol{y}}\in\mathbb{R}^n$ such that $|\vec{\boldsymbol{y}}|=1$. One can always find a rotation matrix that rotates the point $\vec{\boldsymbol{x}}$ to the point $\vec{\boldsymbol{y}}$. Thus, the orbit is $S^{n-1}$, i.e. the set of all unit vectors in $\mathbb{R}^n$.

To determine the little group, we must find the subgroup of SO$(n)$ that leaves $\vec{\boldsymbol{x}}$ invariant. In the case of $\vec{\boldsymbol{x}}=(1,0,0,\ldots,0)$, the little group is the subgroup of SO$(n)$ that leaves the $x_1$-axis fixed, namely SO$(n-1)$. This can be seen explicitly as follows:

$$\left(\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & \mathrm{SO}(n-1) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},$$

where the matrix block obtained by eliminating the first row and column is an SO$(n-1)$ matrix. Thus, using the results of part (b), we conclude that:

$$S^{n-1}\cong\frac{\mathrm{SO}(n)}{\mathrm{SO}(n-1)}\,. \tag{68}$$

**Remark:** In general, the little group $H$ is *not* a normal subgroup of $G$, in which case the manifold $G/H$ is not a Lie group. In light of eq. (68), this last remark is consistent with the theorem mentioned in class that $S^{n-1}$ is not a group manifold for all positive integers $n$ (excluding the special cases of $n=1,2$ and 4).

(d) Prove that $S^{2n-1}$ =U$(n)$/U$(n-1)$ by considering the action of the U$(n)$ matrices on the point $(1,0,0,\ldots,0)\in\mathbb{C}^n$.

Consider the action of U$(n)$ on vectors in $\mathbb{C}^n$. We repeat the analysis give in part (c) almost verbatim. Focus on the point $\vec{\boldsymbol{w}}=(1\,,\,0\,,\,0\,,\,\ldots\,,\,0)\in\mathbb{C}^n$. The orbit through $\vec{\boldsymbol{w}}$ under the action of U$(n)$ consists of all complex vectors $\vec{\boldsymbol{z}}\in\mathbb{C}^n$ such that $|z_1|^2+|z_2|^2+\cdots+|z_n|^2=1$. Writing $z_i=x_i+iy_i$ where $x_i$ and $y_i$ are real numbers, it follows that

$$x_1^2+x_2^2+\cdots+x_n^2+y_1^2+y_2^2+\cdots+y_n^2=1\,.$$

This is the equation for the $(2n-1)$-dimensional sphere, $S^{2n-1}$. To determine the little group, we must find the subgroup of U$(n)$ that leaves $\vec{\boldsymbol{w}}$ invariant. In the case of $\vec{\boldsymbol{w}}=(1,0,0,\ldots,0)$, the little group is the subgroup of U$(n)$ that leaves the complex $w_1$-axis fixed, namely U$(n-1)$. This can be seen explicitly as follows:

$$\left(\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & \mathrm{U}(n-1) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},$$

where the matrix block obtained by eliminating the first row and column is an $\mathrm{U}(n-1)$ matrix. Thus, using the results of part (b), we conclude that:

$$S^{2n-1} \cong \frac{\mathrm{U}(n)}{\mathrm{U}(n-1)} .$$

(e) Complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines in $\mathbb{C}^{n+1}$ through the origin. That is, $\mathbb{CP}^n$ consists of the set of vectors in $\mathbb{C}^{n+1}$ (omitting the zero vector) where we identify $(z_0, z_1, \ldots, z_n) \sim \lambda(z_0, z_1, \ldots, z_n)$, for any nonzero complex number $\lambda$. Without loss of generality, we can restrict our considerations to vectors in $\mathbb{C}^{n+1}$ of modulus 1. Show that $\mathrm{U}(1) \otimes \mathrm{U}(n)$ is the little group of the point $z = (1, 0, 0, \ldots, 0) \in \mathbb{CP}^n$, and that $\mathbb{CP}^n$ is the orbit through $z$ with respect to $\mathrm{U}(n+1)$. Conclude that $\mathbb{CP}^n = \mathrm{U}(n+1)/\mathrm{U}(1) \otimes \mathrm{U}(n)$.

A complex line through the origin is an equivalence relation of points in $\mathbb{C}^{n+1}$ given by:

$$(z_0 \,, z_1 \,, \ldots \,, z_n) \sim \lambda(z_0 \,, z_1 \,, \ldots \,, z_n) \,, \qquad \text{where } \lambda \in \mathbb{C}^* \,, \tag{69}$$

where $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$ is the set of nonzero complex numbers and $(z_0 \,, z_1 \,, \ldots \,, z_n) \in \mathbb{C}^{n+1}$. Without loss of generality, we can restrict our considerations to vectors in $\mathbb{C}^{n+1}$ of modulus 1. This means that we impose the condition

$$\sum_{i=0}^{n} |z_i|^2 = \sum_{i=0}^{n} (x^2 + y^2) = 1 \,, \tag{70}$$

where we have written $z_i = x_i + iy_i$ where $x_i$ and $y_i$ are real numbers. Eq. (70) is an equation for the $(2n+1)$-dimensional sphere, $S^{2n+1}$. When eq. (70) is imposed, the complex number $\lambda$ in eq. (69) must satisfy $|\lambda| = 1$. Hence, $\lambda = e^{i\theta}$, where $0 \leq \theta < 2\pi$, which is in one-to-one correspondence with a one-dimensional circle, $S^1$. Thus, complex projective space $\mathbb{CP}^n$ can also be described as:

$$\mathrm{CP}^n = \{\text{points on } S^{2n+1} \text{ with a circle of phases } S^1 \text{ identified as equivalent}\} .$$

This definition of $\mathrm{CP}^n$ is mathematically written as:

$$\mathrm{CP}^n = S^{2n+1}/S^1 .$$

Consider the action of $\mathrm{U}(n+1)$ on $\mathrm{CP}^n \subset \mathbb{C}^{n+1}$ . Pick the point in $\mathrm{CP}^n$ corresponding to the $(n+1)$ dimensional vector, $z = (1 \,, 0 \,, 0 \,, \ldots \,, 0)$. Note that as a point in $\mathrm{CP}^n$, $z$ is equivalent to all $(n+1)$-dimensional vectors in $\mathbb{C}^{n+1}$ of the form $(e^{i\theta} \,, 0 \,, 0 \,, \ldots \,, 0)$. It is easy to prove that any point in $\mathrm{CP}^n$ can be obtained by the action of some element of $\mathrm{U}(n+1)$ on $z$, since one can always find an element of $\mathrm{U}(n+1)$ to map $z$ to an arbitrary $(n+1)$-dimensional vector of $\mathbb{C}^{n+1}$ of unit modulus.

To determine the little group, consider the following mapping,

$$\left(\begin{array}{c|cccc} e^{i\theta} & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & \mathrm{U}(n) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} e^{i\theta} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{71}$$

Since the following $(n+1)$-dimensional vectors are equivalent,

$$(e^{i\theta}\,,\,0\,,\,0\,,\,\ldots\,,\,0)\;\sim\;(1\,,\,0\,,\,0\,,\,\ldots\,,\,0)\,,$$

eq. (71) implies that the little group consists of all $(n+1)\times(n+1)$ matrices of the form

$$\left(\begin{array}{c|cccc} e^{i\theta} & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & \mathrm{U}(n) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right) \in \mathrm{U}(1)\otimes\mathrm{U}(n)\,, \tag{72}$$

where the matrix block obtained by eliminating the first row and column is a $\mathrm{U}(n)$ matrix. The matrices specified in eq. (72) constitute the group $\mathrm{U}(1)\otimes\mathrm{U}(n)$. Hence, using the results of part (b), it follows that:

$$\mathrm{CP}^n \cong \frac{\mathrm{U}(n+1)}{\mathrm{U}(1)\otimes\mathrm{U}(n)}\,.$$

(f) Real projective space $\mathbb{RP}^n$ can be defined analogously to $\mathbb{CP}^n$ of part (e) by replacing the field of complex numbers with the field of real numbers. What coset space can be identified with $\mathbb{RP}^n$?

A real line through the origin is an equivalence relation of points in $\mathbb{R}^{n+1}$ given by:

$$(x_0\,,\,x_1\,,\,\ldots\,,\,x_n)\;\sim\;\lambda(x_0\,,\,x_1\,,\,\ldots\,,\,x_n)\,,\qquad \text{where } \lambda\in\mathbb{R}^*\,, \tag{73}$$

where $\mathbb{R}^*\equiv\mathbb{R}-\{0\}$ is the set of nonzero real numbers and $(x_0\,,\,x_1\,,\,\ldots\,,\,x_n)\in\mathbb{R}^{n+1}$. Without loss of generality, we can restrict our considerations to vectors in $\mathbb{R}^{n+1}$ of unit length. This means that we impose the condition

$$\sum_{i=0}^{n} x_i^2 = 1\,, \tag{74}$$

which is an equation for the $n$-dimensional sphere, $S^n$. When eq. (74) is imposed, the real number $\lambda$ in eq. (73) must satisfy $\lambda^2=1$. Hence, $\lambda=\pm1$, which is in one-to-one correspondence with the discrete group $\mathbb{Z}_2$ Thus, real projective space $\mathbb{RP}^n$ can also be described as:

$$\mathrm{RP}^n = \{\text{points on } S^n \text{ with antipodal points identified as equivalent}\}\,.$$

This definition of $\mathrm{RP}^n$ is mathematically written as:

$$\mathrm{RP}^n \cong S^n/\mathbb{Z}_2\,. \tag{75}$$

Consider the action of $\mathrm{O}(n+1)$ on $\mathrm{RP}^n\subset\mathbb{R}^{n+1}$ . Pick the point in $\mathrm{RP}^n$ corresponding to the $(n+1)$-dimensional vector, $\vec{\boldsymbol{x}}=(1\,,\,0\,,\,0\,,\,\ldots\,,\,0)$. Note that as a point in $\mathrm{RP}^n$, $\vec{\boldsymbol{x}}$

is equivalent $(-1\,,\,0\,,\,0\,,\,\ldots\,,\,0)$. It is easy to prove that any point in $\mathrm{RP}^n$ can be obtained by the action of some element of $\mathrm{O}(n+1)$ on $\vec{\boldsymbol{x}}$, since one can always find an element of $\mathrm{O}(n+1)$ to map $\vec{\boldsymbol{x}}$ to an arbitrary $(n+1)$-dimensional vector of $\mathbb{R}^{n+1}$ of unit length.

To determine the little group, consider the following mapping,

$$\left(\begin{array}{c|cccc} \pm 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & \mathrm{O}(n) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\,. \tag{76}$$

Since $(1\,,\,0\,,\,0\,,\,\ldots\,,\,0) \sim (-1\,,\,0\,,\,0\,,\,\ldots\,,\,0)$, the little group consists of all $(n+1)\times(n+1)$ matrices of the form

$$\left(\begin{array}{c|cccc} \pm 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & \mathrm{O}(n) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right) \in \mathrm{O}(1)\otimes\mathrm{O}(n)\,, \tag{77}$$

where the matrix block obtained by eliminating the first row and column is an $\mathrm{O}(n)$ matrix. Note that the orthogonal group, $\mathrm{O}(1)=\{+1,-1\}$, which implies that the matrices of eq. (77) constitute the group $\mathrm{O}(1)\otimes\mathrm{O}(n)$. Hence, using the results of part (b), it follows that:

$$\mathrm{RP}^n \cong \frac{\mathrm{O}(n+1)}{\mathrm{O}(1)\otimes\mathrm{O}(n)}\,. \tag{78}$$

**<u>Remark:</u>** By a computation that is nearly identical to the one given in part (c), one can prove that $S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n)$. Hence, it follows from eq. (75) that:

$$\mathrm{RP}^n \cong \frac{\mathrm{O}(n+1)/\mathrm{O}(n)}{\mathbb{Z}_2}\,.$$

It is tempting to claim that this result immediately implies eq. (78) in light of the fact that $\mathrm{O}(1)\cong\mathbb{Z}_2$, but this last step is not well-defined until one specifies the nature of the equivalence relation implicit in eq. (78) as well as the precise embedding of the groups. Thus, one cannot use this shortcut to derive eq. (78).

(g) In parts (c)–(f), check that $\dim(G/H) = \dim G - \dim H$.

In part (c), dim $S^{n-1} = n-1$, dim $\mathrm{SO}(n) = \frac{1}{2}n(n-1)$ and dim $\mathrm{SO}(n-1) = (n-1)^2$. Hence, we check that

$$\tfrac{1}{2}n(n-1) - \tfrac{1}{2}(n-1)(n-2) = \tfrac{1}{2}(n-1)[n-(n-2)] = n-1\,. \quad \checkmark$$

In part (d), dim $S^{2n-1} = 2n-1$, dim $\mathrm{U}(n) = n^2$ and dim $\mathrm{U}(n-1) = \frac{1}{2}(n-1)(n-2)$. Hence, we check that

$$n^2 - (n-1)^2 = 2n-1\,. \quad \checkmark$$

In part (e), dim $\mathbb{CP}^n = 2n$, dim $\mathrm{U}(n+1) = (n+1)^2$ and dim $\mathrm{U}(n)\otimes\mathrm{U}(n) = 1+n^2$. Hence, we check that

$$(n+1)^2 - 1 - n^2 = 2n\,. \quad \checkmark$$

In part (f), dim $\mathbb{RP}^n = n$, dim $\mathrm{O}(n+1) = \frac{1}{2}n(n+1)$ and dim $\mathrm{O}(1)\otimes\mathrm{O}(n) = \frac{1}{2}n(n-1)$, since the discrete group $\mathrm{O}(1)\cong\mathbb{Z}_2$ does not contribute the dimension. Hence, we check that

$$\tfrac{1}{2}n(n+1) - \tfrac{1}{2}n(n-1) = n\,. \quad \checkmark$$

In all cases above, dim refers to *real* dimensions (sometimes denoted by $\dim_{\mathbb{R}}$). In particular, since $\mathrm{U}(n+1)$ is *not* a complex manifold, it does not make sense to employ complex dimensions in the case of part (e). Note that $\mathbb{CP}^n$ is a complex manifold of $n$ complex dimensions, so that $\dim_{\mathbb{R}}\mathbb{CP}^n = 2n$ as stated above.

(h) $\mathbb{CP}^n$ is a manifold of $n$ complex (or $2n$ real) dimensions. $\mathbb{CP}^1$ is homeomorphic to which well-known two-dimensional real manifold?

$\mathbb{CP}^1$ consists of all complex 2-vectors such that $(z_0\,,\,z_1)\sim\lambda(z_0\,,\,z_1)$ for all $\lambda\in\mathbb{C}^*$. Thus, we can express $\mathbb{CP}^1$ as the union of two sets:

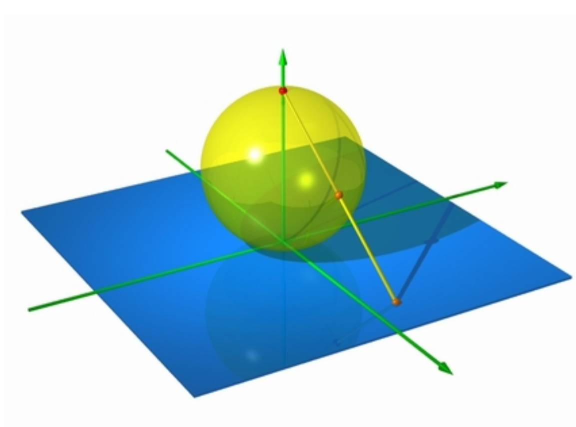
$$\{\text{all vectors equivalent to } (z_0\,,\,z_1)\,,\ z_0\neq 0\}\cup\{\text{all vectors equivalent to } (0\,,\,z_1)\,,z_1\neq 0\}\,.$$

If $z_0\neq 0$, then $(z_0\,,\,z_1)\sim(1\,,\,z_1/z_0)\equiv(1\,,\,z)$, where $z$ ranges over all numbers in the finite complex plane (including $z=0$). Likewise, $(0\,,\,z_1)\sim(0,1)$ for all $z_1\in\mathbb{C}^*$. Hence, we conclude that:

$$\mathbb{CP}^1\cong\mathbb{C}\cup\{(0,1)\}\,.$$

In words, $\mathbb{CP}^1$ is isomorphic to the union of the finite complex plane and a single point.

This should remind you of the representation of the extended complex plane as the union of the finite complex plane and the point of infinity. This representation is topologically equivalent to the stereographic projection of the two-dimensional sphere (called the Riemann sphere). In particular, we represent $\mathbb{C}\cong\mathbb{R}^2\cong S^2-\{\text{north pole}\}$ by the mapping illustrated in the figure below.



The south pole of the Riemann sphere is tangent to the complex plane. A straight line is drawn from the north pole (which lies above the complex plane) to the complex plane. This line intersects the sphere (i.e. the surface of the ball) once and intersects the complex plane once.

This establishes a one-to-one correspondence between the points of the finite complex plane and $S^2 - \{\text{north pole}\}$. The north pole itself maps onto the point of infinity. Hence,

$$S^2 \cong \mathbb{C} \cup \{\text{point of infinity}\}\,,$$

as a topological homeomorphism. Thus, using the stereographic projection, we have a one-to-one onto map between $\mathbb{CP}^1$ and the Riemann sphere. It follows that:

$$\mathbb{CP}^1 \cong S^2\,.$$

That is, one can introduce a complex structure on the two-dimensional sphere and realize it as a complex manifold of one complex dimension. Indeed, this manifold is $\mathbb{CP}^1$.

5. Let $A$ be an even-dimensional complex antisymmetric $2n\times 2n$ matrix, where $n$ is a positive integer. We define the *pfaffian* of $A$, denoted by $\operatorname{pf} A$, by:

$$\operatorname{pf} A = \frac{1}{2^n n!}\sum_{p\in S_{2n}} (-1)^p\, A_{i_1 i_2} A_{i_3 i_4}\cdots A_{i_{2n-1} i_{2n}}\,, \qquad (79)$$

where the sum is taken over all permutations

$$p = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ i_1 & i_2 & \cdots & i_{2n}\end{pmatrix}$$

and $(-1)^p$ is the sign of the permutation $p \in S_{2n}$. If $A$ is an odd-dimensional complex antisymmetric matrix, the corresponding pfaffian is defined to be zero.

(a) By explicit calculation, show that[10]

$$\det A = (\operatorname{pf} A)^2\,, \qquad (80)$$

for any $2\times 2$ and $4\times 4$ complex antisymmetric matrix $A$.

The most general $2\times 2$ antisymmetric matrix is:

$$A = \begin{pmatrix} 0 & a \\ -a & 0\end{pmatrix}\,.$$

By definition of the pfaffian given in eq. (79),

$$\operatorname{pf} A = \tfrac{1}{2}[a - (-a)] = a\,,$$

and $\det A = a^2$. Hence, $\det A = (\operatorname{pf} A)^2$. ✓

[10]Eq. (80) holds for all complex antisymmetric $2n \times 2n$ matrices, where $n$ is any positive number. A general proof is provided in a class handout entitled, *Antisymmetric Matrices and the Pfaffian.*

The most general $4 \times 4$ antisymmetric matrix is:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} .$$

Consider the 24 elements of the permutation group $S_4$. Using the definition of the pfaffian given in eq. (79), three of the terms that appear in the sum are:

$$\text{pf}\, A = \tfrac{1}{8}\left[a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} + \cdots\right] , \tag{81}$$

where we have noted that $\left(\begin{smallmatrix}1&2&3&4\\1&3&2&4\end{smallmatrix}\right)$ is an odd permutation and $\left(\begin{smallmatrix}1&2&3&4\\1&4&2&3\end{smallmatrix}\right)$ is an even permutation. It is clear that the other 21 terms that appear in the sum just reproduce the same three terms displayed in eq. (81) with the same signs. Hence, we conclude that

$$\text{pf}\, A = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} .$$

The computation of the determinant of $A$ is straightforward.

$$\det A = -a_{12}\det\begin{pmatrix} -a_{12} & a_{23} & a_{24} \\ -a_{13} & 0 & a_{34} \\ -a_{14} & -a_{34} & 0 \end{pmatrix} + a_{13}\det\begin{pmatrix} -a_{12} & 0 & a_{24} \\ -a_{13} & -a_{23} & a_{34} \\ -a_{14} & -a_{24} & 0 \end{pmatrix} - a_{14}\det\begin{pmatrix} -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \\ -a_{14} & -a_{24} & -a_{34} \end{pmatrix}$$

$$\begin{aligned} &= -a_{12}a_{24}a_{13}a_{34} + a_{12}a_{34}(a_{12}a_{34} + a_{23}a_{14}) - a_{13}a_{12}a_{24}a_{34} + a_{13}a_{24}(a_{13}a_{24} - a_{14}a_{23}) \\ &\qquad + a_{14}a_{12}a_{23}a_{34} - a_{14}a_{23}(a_{13}a_{24} - a_{14}a_{23}) \\ &= a_{14}^2a_{23}^2 + a_{13}^2a_{24}^2 + a_{12}^2a_{34}^2 - 2a_{12}a_{13}a_{24}a_{34} + 2a_{12}a_{14}a_{23}a_{34} - 2a_{13}a_{14}a_{23}a_{24} \\ &= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 . \end{aligned}$$

Once again, we see that $\det A = (\text{pf}\, A)^2$. ✓

(b) Prove that the determinant of any odd-dimensional complex antisymmetric matrix vanishes. As a result, the definition of the pfaffian in the odd-dimensional case is consistent with the result of eq. (80).

Starting with $A = -A^{\mathsf{T}}$ and taking the determinant of both sides of this equation,

$$\det A = \det(-A^{\mathsf{T}}) = (-1)^n \det A^{\mathsf{T}} = (-1)^n \det A ,$$

where $n$ is the dimension of the matrix $A$. If $n$ is odd, then $(-1)^n = -1$, in which case,

$$\det A = -\det A ,$$

from which we conclude that $\det A = 0$.

(c) Given an arbitrary $2n\times 2n$ complex matrix $B$ and complex antisymmetric $2n\times 2n$ matrix $A$, use eq. (80) to prove the following identity:

$$\text{pf}\,(BAB^{\mathsf T}) = \text{pf}\,A\,\det B\,. \tag{82}$$

Using eq. (79), if follows that $\text{pf}\,A = (2^n n!)^{-1}\epsilon_{i_1 i_2\cdots i_{2n-1}i_{2n}}A_{i_1i_2}A_{i_3i_4}\cdots A_{i_{2n-1}i_{2n}}$. Hence,

$$\begin{aligned}\text{pf}\,(BAB^{\mathsf T}) &= \frac{1}{2^n n!}\epsilon_{i_1j_1i_2j_2\cdots i_nj_n}(B_{i_1k_1}A_{k_1\ell_1}B_{j_1\ell_1})(B_{i_2k_2}A_{k_2\ell_2}B_{j_2\ell_2})\cdots(B_{i_nk_n}A_{k_n\ell_n}B_{j_n\ell_n})\\ &= \frac{1}{2^n n!}\epsilon_{i_1j_1i_2j_2\cdots i_nj_n}B_{i_1k_1}B_{j_1\ell_1}B_{i_2k_2}B_{j_2\ell_2}\cdots B_{i_nk_n}B_{j_n\ell_n}A_{k_1\ell_1}A_{k_2\ell_2}\cdots A_{k_n\ell_n}\,,\end{aligned}$$

after rearranging the order of the matrix elements of $A$ and $B$. We recognize the definition of the determinant of a $2n\times 2n$-dimensional matrix,

$$\det B\,\epsilon_{k_1\ell_1k_2\ell_2\cdots k_n\ell_n} = \epsilon_{i_1j_1i_2j_2\cdots i_nj_n}B_{i_1k_1}B_{j_1\ell_1}B_{i_2k_2}B_{j_2\ell_2}\cdots B_{i_nk_n}B_{j_n\ell_n}\,. \tag{83}$$

Inserting eq. (83) into the expression for $\text{pf}\,(BAB^{\mathsf T})$ yields

$$\text{pf}\,(BAB^{\mathsf T}) = \frac{1}{2^n n!}\,\det B\,\epsilon_{k_1\ell_1k_2\ell_2\cdots k_n\ell_n}A_{k_1\ell_1}A_{k_2\ell_2}\cdots A_{k_n\ell_n} = \text{pf}\,A\,\det B\,.$$

(d) A complex $2n\times 2n$ matrix $S$ is called *symplectic* if $S^{\mathsf T}JS = J$, where $S^{\mathsf T}$ is the transpose of $S$ and

$$J \equiv \begin{pmatrix} \mathbb{0}_n & \mathbb{1}_n \\ -\mathbb{1}_n & \mathbb{0}_n\end{pmatrix}\,, \tag{84}$$

where $\mathbb{1}_n$ is the $n\times n$ identity matrix and $\mathbb{0}_n$ is the $n\times n$ zero matrix. Prove that the set of $2n\times 2n$ complex symplectic matrices, denoted by $\text{Sp}(n,\mathbb{C})$, is a matrix Lie group[11] [*i.e.*, it is a topologically closed subgroup of $\text{GL}(2n,\mathbb{C})$].

Consider the set,

$$\text{Sp}(n,\mathbb{C}) = \{S\in \text{GL}(2n,\mathbb{C})\,|\,S^{\mathsf T}JS = J\}\,,$$

where $J$ is defined in eq. (84). To prove that this is a subgroup of $\text{GL}(2n,\mathbb{C})$, we first verify the closure property. If $S_1^{\mathsf T}JS_1 = J$ and $S_2^{\mathsf T}JS_2 = J$, then

$$(S_1S_2)^{\mathsf T}J(S_1S_2) = S_2^{\mathsf T}S_1^{\mathsf T}JS_1S_2 = S_2^{\mathsf T}JS_2 = J\,. \quad \checkmark$$

Next, we note that the identity matrix $\mathbb{1}_{2n}\in \text{GL}(2n,\mathbb{C})$ is an element of $\text{Sp}(n,\mathbb{C})$, since

$$\mathbb{1}_{2n}^{\mathsf T}J\mathbb{1}_{2n} = J\,. \quad \checkmark$$

Finally, we check that if $S\in\text{Sp}(n,\mathbb{C})$ then $S^{-1}\in\text{Sp}(n,\mathbb{C})$, by verifying that

$$(S^{-1})^{\mathsf T}JS^{-1} = (S^{-1})^{\mathsf T}(S^{\mathsf T}JS)S^{-1} = J\,,$$

where we have used $S^{\mathsf T}JS = J$. $\checkmark$

[11]Warning: many authors denote the group of $2n\times 2n$ complex symplectic matrices by $\text{Sp}(2n,\mathbb{C})$.

Note that $\mathrm{Sp}(n,\mathbb{C})$ is a closed subgroup of $\mathrm{GL}(2n,\mathbb{C})$ by virtue of the theorem given in class that states if $f$ and $g$ are two functions that map Hausdorff spaces $X \longrightarrow Y$, then the set $\{x \in X \mid f(x) = g(x)\}$ is closed. We apply this theorem to the current problem by choosing $f(S) = S^{\mathsf{T}}JS$ and $g(S) = J$.

(e) Prove that if $S$ is a symplectic matrix, then $\det S = 1$.

Since $S$ is symplectic, $J = S^{\mathsf{T}}JS$. Taking the determinant of this equation and noting that $\det(S^{\mathsf{T}}) = \det S$, it follows that,

$$\det J = \det(S^{\mathsf{T}}JS) = [\det S]^2 \det J\,. \tag{85}$$

A simple computation gives $\det J = 1$. Hence, eq. (85) implies that $[\det S]^2 = 1$, and it follows that $\det S = \pm 1$.

To prove that $\det S = 1$ for all $S \in \mathrm{Sp}(n,\mathbb{C})$, we shall make use of part (c). In particular, using $\det(S^{\mathsf{T}}) = \det S$ and eq. (82),

$$\mathrm{pf}(S^{\mathsf{T}}JS) = \mathrm{pf}\, J \det S\,. \tag{86}$$

Since $S$ is symplectic, $S^{\mathsf{T}}JS = J$, which when inserted into eq. (86) gives

$$\mathrm{pf}\, J = \mathrm{pf}\, J \det S\,. \tag{87}$$

Using the definition of the pfaffian given in eq. (79), it follows that $\mathrm{pf}\, J = 1$. Thus, it is permissible to divide eq. (87) by $\mathrm{pf}\, J$, which immediately yields

$$\det S = 1\,,$$

for all complex symplectic matrices $S$.

**First alternative derivation**. Another way to show that there are no symplectic matrices with $\det S = -1$ is to prove that $\mathrm{Sp}(n,\mathbb{C})$ is a connected group.[12] Thus, all elements of $\mathrm{Sp}(n,\mathbb{C})$ are continuously connected to the identity element. Since $\mathbb{1}_{2n} \in \mathrm{Sp}(n,\mathbb{C})$ is a matrix with determinant equal to 1, it follows by continuity that all elements of $\mathrm{Sp}(n,\mathbb{C})$ must have unit determinant.

**Second alternative derivation**. In Anthony Zee, *Group Theory in a Nutshell for Physicists*, another proof that the determinant of a symplectic matrix is equal to one is provided. However, this proof provided is not complete. Here, I will provide Zee's argument and then show how to complete the proof.

Zee examines the characteristic polynomial, $P(z)$, of the symplectic matrix $S$, which can be manipulated as follows.

$$P(z) = \det(S - z\mathbb{1}_{2n}) = z^{2n}\det(z^{-1}S - \mathbb{1}_{2n}) = z^{2n}\det S \det(z^{-1}\mathbb{1}_{2n} - S^{-1})\,. \tag{88}$$

[12] The proof is a generalization of the proof that $\mathrm{SL}(2,\mathbb{C})$ is simply connected given in part (b) of problem 2. In this proof, one shows that $\mathrm{Sp}(n,\mathbb{C})$ is topologically equivalent to $\mathrm{U}(n) \times \mathbb{R}^{n(n+1)}$, which is a connected space. We do not present this proof here; for further details, see e.g. Chapter 10 of Denis Serre, *Matrices: Theory and Applications*, Second Edition (Springer Science, New York, 2010).

Since $S$ is symplectic, we have $J = S^{\mathsf{T}} J S$, which implies that $S^{-1} = -JS^{\mathsf{T}}J$, after making use of $J^{-1} = -J$. Hence, it follows that

$$P(z) = z^{2n} \det S \det(z^{-1}\mathbb{1}_{2n} + JS^T J) = z^{2n} (\det J)^2 \det S \det(S^{\mathsf{T}} - z^{-1}\mathbb{1}_{2n}) \,.$$

As previously noted, $\det J = 1$. Hence,

$$P(z) = z^{2n} \det S \det\bigl[(S - z^{-1}\mathbb{1}_{2n})^{\mathsf{T}}\bigr] = z^{2n} \det S \det(S - z^{-1}\mathbb{1}_{2n}) = z^{2n}(\det S) P(z^{-1}) \,. \quad (89)$$

That is, we have proved that the characteristic equation for a symplectic matrix $S$ satisfies,

$$P(z) = z^{2n}(\det S) P(z^{-1}) \,. \quad (90)$$

The eigenvalues of $S$ are obtained by solving the equation $P(z) = 0$. Since $S$ is invertible, $S$ does not possess any zero eigenvalues. Hence, $P(z) = 0$ implies that $P(z^{-1}) = 0$. We conclude that if $\lambda$ is an eigenvalue of $S$ then $\lambda^{-1}$ is also an eigenvalue of $S$. That is, the eigenvalues of $S$ come in pairs, $\{\lambda\,,\, \lambda^{-1}\}$. Moreover, if $\lambda \neq \pm 1$, then it follows that the multiplicity of the eigenvalues $\lambda$ and $\lambda^{-1}$ are the same. This latter result follows after taking $r$ derivatives of eq. (90) with respect to $z$, where $r$ is the multiplicity of the eigenvalue $\lambda$, and then setting $z = \lambda$ .

What happens when $\lambda = \lambda^{-1}$ or equivalently when $\lambda = \pm 1$? Zee does not consider this possibility, but it is critical for the argument, because the determinant of $S$ is a product of its eigenvalues. It then follows that

$$\det S = (-1)^d \,, \quad (91)$$

where $d$ is the multiplicity of the eigenvalue $-1$. We shall now argue that $d$ must be even. Suppose that $-1$ is an eigenvalue of $S$, in which case $\mathbb{1}_{2n} + S$ is not invertible. However, it is possible to perturb the matrix $S \to S_\epsilon$ in such a way that it remains symplectic such that $\mathbb{1}_{2n} + S_\epsilon$ is invertible, implying that $-1$ is not an eigenvalue of $S_\epsilon$.[13] If the perturbation is small, then $S_\epsilon$ will possess an eigenvalue $\lambda$ with multiplicity $r$ that is very close to $-1$. But, using the results obtained above, $S_\epsilon$ will possess a second eigenvalue $\lambda^{-1}$ with the same multiplicity $r$ that is also very close to $-1$. In the limit of $\epsilon \to 0$ where the perturbation is removed, the eigenvalues $\lambda$ and $\lambda^{-1}$ coalesce, resulting in a degenerate eigenvalue $-1$ with multiplicity $2r$.[14] We conclude that $d = 2r$ in eq. (91), and it follows that $\det S = 1$.

(f) Using the results of parts (d) and (e), prove that the matrix Lie groups $\mathrm{Sp}(1, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$ are isomorphic.

Consider an arbitrary complex $2 \times 2$ matrix,

$$S \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,,$$

[13] Details are given in Israel Gohberg, Peter Lancaster and Leiba Rodman, *Indefinite Linear Algebra and Applications* (Birkhäuser,Basel, Switzerland, 2005) pp. 64–66. In particular, see Lemma 4.8.6 of this reference, which was inspired by an appendix on symplectic matrices that appears in Peter D. Lax, *Linear Algebra and its Applications*, 2nd edition (John Wiley & Sons, Inc., Hoboken, NJ, 2007) pp. 308–312.

[14] By a similar argument, the multiplicity of the eigenvalue $+1$ must also be even.

where $a, b, c, d \in \mathbb{C}$. Using the result of part (e), if $S$ is a symplectic matrix then $\det S = 1$, which implies that $S \in \mathrm{SL}(2,\mathbb{C})$. Since $\mathrm{Sp}(1,\mathbb{C})$ is a group, we can conclude that it is a subgroup of $\mathrm{SL}(2,\mathbb{C})$. That is,

$$\mathrm{Sp}(1,\mathbb{C}) \leq \mathrm{SL}(2,\mathbb{C})\,. \tag{92}$$

Next, if $S \in \mathrm{SL}(2,\mathbb{C})$ then $ad - bc = 1$. We now compute $S^{\mathsf{T}}JS$,

$$S^{\mathsf{T}}JS = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & ad-bc \\ bc-ad & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J\,.$$

Hence, $S^{\mathsf{T}}JS = J$, which implies that $S \in \mathrm{Sp}(1,\mathbb{C})$. Since $\mathrm{SL}(2,\mathbb{C})$ is a group, we can conclude that it is a subgroup of $\mathrm{Sp}(1,\mathbb{C})$. That is,

$$\mathrm{SL}(2,\mathbb{C}) \leq \mathrm{Sp}(1,\mathbb{C}). \tag{93}$$

Eqs. (92) and (93) taken ogether imply that

$$\mathrm{Sp}(1,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C})\,.$$

For $2\times 2$ symplectic matrices, the determinental constraint is the *only* constraint imposed on a complex $2\times 2$ matrix. For values of $n > 1$, the condition $S^{\mathsf{T}}JS = J$ imposes additional constraints beyond that of $\det S = 1$.

**Remark:** For $n = 2$, one can prove the isomorphism $\mathrm{Sp}(2,\mathbb{C}) \cong \mathrm{SO}(5,\mathbb{C})$. For values of $n \geq 3$, $\mathrm{Sp}(n,\mathbb{C})$ is not isomorphic to any of the other classical groups, and thus must be considered as an independent family of simple Lie groups.

6. The two-dimensional Poincaré group P(2) is the group consisting of two-dimensional Lorentz transformations [i.e., transformations on 2-vectors $\left(\begin{smallmatrix} ct \\ x \end{smallmatrix}\right)$ that preserve $x^2 - c^2t^2$] and translations in time and space. P(2) can be represented by $3\times 3$ matrices acting linearly on the column vector, $\left(\begin{smallmatrix} ct \\ x \\ 1 \end{smallmatrix}\right)$, in analogy with the two-dimensional Euclidean group, E(2), worked out in class.

The two-dimensional Poincaré group P(2) is the group consisting of two-dimensional Lorentz transformations and spacetime translations. The most general two-dimensional Lorentz transformation (which preserves the quantity $x^2 - c^2t^2$) can be written in the following form:

$$\begin{pmatrix} ct \\ x \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}\,, \tag{94}$$

where[15]

$$\cosh\xi \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma\,, \qquad \sinh\xi = \frac{\gamma v}{c}\,.$$

[15]The parameter $\xi$ (which is defined as $\tanh\xi \equiv v/c$, where $v$ is the velocity) is called the *rapidity*. The parametrization of the Lorentz transformation given in eq. (94) is convenient since the hyperbolic trigonometric identity, $\cosh^2\xi - \sinh^2\xi = 1$, ensures that $x^2 - c^2t^2$ is invariant under Lorentz transformations. Note that $\left(\begin{smallmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{smallmatrix}\right)$ is the most general element of the group SO(1,1).

To incorporate spacetime translations $x \to x + x_0$ and $t \to t + t_0$, we can employ $3 \times 3$ matrices,

$$\begin{pmatrix} ct \\ x \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh\xi & \sinh\xi & ct_0 \\ \sinh\xi & \cosh\xi & x_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ 1 \end{pmatrix} . \tag{95}$$

That is, the most general element of the two-dimensional Poincaré group P(2) is given by:

$$\begin{pmatrix} \cosh\xi & \sinh\xi & ct_0 \\ \sinh\xi & \cosh\xi & x_0 \\ 0 & 0 & 1 \end{pmatrix} .$$

(a) Find the infinitesimal generators (i.e., differential operators) of the corresponding Lie algebra, $\mathfrak{p}(2)$. Work out the commutation relations of $\mathfrak{p}(2)$.

Consider a Lie transformation group $G$ that acts on a manifold $M$ from the left. We define

$$x'^{\,i} = \Phi^i(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})\,, \tag{96}$$

where $A \equiv (a_1, a_2, \ldots a_n) \in G$ acts on $\vec{\boldsymbol{x}} \in M$. Following section 5 of the class handout entitled, *Local properties of a Lie group*, the generators of an infinitesimal Lie group transformation consist of the differential operators,

$$X_k(\vec{\boldsymbol{x}}) \equiv -u^i_k(\vec{\boldsymbol{x}})\frac{\partial}{\partial x^i}\,, \tag{97}$$

where

$$u^i_k(\vec{\boldsymbol{x}}) \equiv \left(\frac{\partial\Phi^i(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})}{\partial a^k}\right)_{\vec{\boldsymbol{a}}=0} . \tag{98}$$

As shown in the class handout cited above, the infinitesimal generators satisfy the same commutations as the corresponding Lie algebra,

$$[X_i, X_j] = f_{ij}^{k} X_k\,,$$

where this equation should be interpreted as an operator equation that acts on a function.

To compute the infinitesimal generators of P(2), we consider eq. (95) for infinitesimal $(ct_0, x_0, \xi)$, where $\sinh\xi \simeq \xi$ and $\cosh\xi \simeq 1$ to first order in $\xi$. Then to evaluate eq. (98), we identify $\vec{\boldsymbol{a}} = (ct_0, x_0, \xi)$ and $\vec{\boldsymbol{x}} = (ct, x, 1)$. In particular, $\Phi(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}})$ in eq. (98) is given by

$$\Phi(\vec{\boldsymbol{a}}\,;\,\vec{\boldsymbol{x}}) = \begin{pmatrix} 1 & \xi & ct_0 \\ \xi & 1 & x_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} ct + \xi x + ct_0 \\ \xi ct + x + x_0 \\ 1 \end{pmatrix} .$$

Thus, $u^i_k(\vec{\boldsymbol{x}})$ is easily evaluated and we obtain:

$$\begin{aligned} &u^1_1 = 1\,, \qquad && u^1_2 = 0\,, \qquad && u^1_3 = x\,, \\ &u^2_1 = 0\,, && u^2_2 = 1\,, && u^2_3 = ct\,. \end{aligned}$$

Then, eq. (97) yields the infinitesimal generators,

$$X_1 = -\frac{1}{c}\frac{\partial}{\partial t}\,, \qquad X_2 = -\frac{\partial}{\partial x}\,, \qquad X_3 = -\frac{x}{c}\frac{\partial}{\partial t} - ct\frac{\partial}{\partial x}\,.$$

The commutation relations are easily evaluated:

$$[X_1, X_2] = \frac{1}{c}\left(\frac{\partial^2}{\partial t\partial x} - \frac{\partial^2}{\partial x\partial t}\right) = 0$$

$$[X_1, X_3] = \frac{x}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{c}\frac{\partial}{\partial t}\left(-ct\frac{\partial}{\partial x}\right) - \frac{x}{c^2}\frac{\partial^2}{\partial t^2} + ct\frac{\partial}{\partial x}\left(-\frac{1}{c}\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial x} = -X_2\,.$$

$$[X_2, X_3] = ct\frac{\partial^2}{\partial x^2} + \frac{1}{c}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial t}\right) - ct\frac{\partial^2}{\partial x^2} - \frac{x}{c}\frac{\partial^2}{\partial x\partial t} = \frac{1}{c}\frac{\partial}{\partial t} = -X_1\,,$$

where we have assumed that the infinitesimal generators are acting on well-behaved functions so that the mixed second partial derivatives are equal. Thus, we have established that:

$$[X_1, X_2] = 0\,, \qquad [X_1, X_3] = -X_2\,, \qquad [X_2, X_3] = -X_1\,.$$

As a check, we can compute the commutation relations of the Lie algebra $\mathfrak{p}(2)$ by expanding the P(2) transformation to first order in the group parameters,

$$\begin{pmatrix} 1 & \xi & ct_0 \\ \xi & 1 & x_0 \\ 0 & 0 & 1 \end{pmatrix} \simeq \mathbf{I} + ct_0\mathcal{A}_1 + x_0\mathcal{A}_2 + \xi\mathcal{A}_3\,,$$

where

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to verify by matrix multiplication that

$$[\mathcal{A}_1, \mathcal{A}_2] = 0\,, \qquad [\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_2\,, \qquad [\mathcal{A}_2, \mathcal{A}_3] = -\mathcal{A}_1\,. \tag{99}$$

Thus, we have confirmed that the commutation relations satisfied by the infinitesimal generators are isomorphic to the Lie algebra $\mathfrak{p}(2)$, as expected.

(b) Compute the Cartan-Killing form. Show that P(2) is noncompact and non-semisimple.

The Cartan-Killing form is defined in terms of the Cartan metric tensor, $g_{ij} \equiv f_{ik}^{\ell} f_{j\ell}^{k}$, where the $f_{ij}^{k}$ are the structure constants of the Lie algebra, and there is an implicit sum over the repeated indices $k$ and $\ell$. Using eq. (99), we see that the only nonzero structure constants are:

$$f_{13}^{2} = f_{23}^{1} = -f_{31}^{2} = -f_{32}^{1} = -1\,.$$

Hence, it follows that only one element of the Cartan metric tensor is nonzero,

$$g_{33} = f_{31}^{2} f_{32}^{1} + f_{32}^{1} f_{31}^{2} = 2 \,.$$

That is,

$$g_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \,.$$

Since $g_{ij}$ is degenerate (i.e., $\det g = 0$), it follows that P(2) is non-semisimple. To see that P(2) is non-compact, simply note that the coordinate $\xi$ on the P(2) group manifold is unbounded.

(c) Express the Lie algebra $\mathfrak{p}(2)$ as a semidirect sum of two abelian subalgebras.

Eq. (99) exhibits the structure of a semidirect sum. Note that $\mathcal{A}_3$ generates the Lie algebra of the two-dimensional Lorentz group, $\mathfrak{so}(1,1)$,[16] and $\mathcal{A}_1$, $\mathcal{A}_2$ generate the Lie algebra of the two-dimensional group of spacetime translations, $\mathfrak{t}(2)$. In particular, $\mathfrak{t}(2)$ is an invariant subalgebra (or ideal) of $\mathfrak{p}(2)$, since for $\mathcal{B} \in \mathfrak{t}(2)$ and $\mathcal{A} \in \mathfrak{p}(2)$ we have $[\mathcal{B}, \mathcal{A}] \in \mathfrak{t}(2)$ [which implies that $\mathfrak{so}(1,1) \cong \mathfrak{p}(2)/\mathfrak{t}(2)$]. In contrast, $\mathfrak{so}(1,1)$ is not an invariant subalgebra of $\mathfrak{p}(2)$. Hence, $\mathfrak{p}(2)$ is the semidirect sum of these two groups. That is, $\mathfrak{p}(2) \cong \mathfrak{t}(2) \ni\!\!\!+ \mathfrak{so}(1,1)$.

7. (a) Show that the Lie algebra of U($n$) can be written as a direct sum, $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1)$.

The Lie algebra of U($n$) can be written as a direct sum, $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1)$. To verify this claim, we can make use of eqs. (1)–(3) in the class handout entitled *Properties of the Gell-Mann matrices.* Consider the $n^2$ generators,

$$(E_\ell^k)_{ij} = \delta_{\ell i} \delta_{kj} \,, \tag{100}$$

which satisfy the following commutation relations (as is easily verified),

$$\big[E_\ell^k \,,\, E_n^m\big] = \delta_n^k E_\ell^m - \delta_\ell^m E_n^k \,. \tag{101}$$

The matrices $E_\ell^k$ also satisfy the hermiticity condition,

$$(E_\ell^k)^\dagger = E_k^\ell \,. \tag{102}$$

Thus, we can use the $E_\ell^k$ to construct the $n^2$ hermitian matrix generators (using the physicist's convention) of $\mathfrak{u}(n)$ by employing suitable linear combinations. The corresponding off-diagonal hermitian generators are of the form $E_\ell^k + E_k^\ell$ and $-i(E_\ell^k - E_k^\ell)$ in analogy with the off-diagonal Gell-Mann matrices. There are $n$ diagonal generators, $E_\ell^\ell$ ($\ell = 1, 2, \ldots, n$),

[16]Note that SO(1,1)$\cong \mathbb{R}$, via the map $\begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \mapsto \xi$, and SO(2)$\cong$U(1)$\cong \mathbb{R}/\mathbb{Z} \cong$SO(1,1)$/\mathbb{Z}$, where $\mathbb{R}$ is the group of real numbers under addition. Hence, $\mathfrak{so}(1,1) \cong \mathfrak{so}(2) \cong \mathfrak{u}(1)$. Indeed, all one-dimensional real Lie algebras are isomorphic.

consisting of one non-zero entry occupying the $\ell\ell$ element of the matrix. Note that

$$\sum_\ell E^\ell_\ell = \mathbf{I}\,,$$

where $\mathbf{I}$ is the $n\times n$ identity matrix. We can identify the traceless generators of $\mathfrak{su}(n)$ by defining

$$(F^k_\ell)_{ij} \equiv (E^k_\ell)_{ij} - \frac{1}{n}\delta_{k\ell}\delta_{ij}\,. \tag{103}$$

The off-diagonal generators of $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ coincide. Since,

$$\sum_\ell F^\ell_\ell = 0\,, \tag{104}$$

it follows that there are only $n-1$ linearly independent diagonal generators of $\mathfrak{su}(n)$. The $F^k_\ell$ also satisfy the same commutation relations as the $E^k_\ell$ [cf. eq. (101)],

$$\left[F^k_\ell\,,\,F^m_n\right] = \delta^k_n F^m_\ell - \delta^m_\ell F^k_n\,. \tag{105}$$

The $(n-1)$ linearly independent diagonal generators chosen from the $F^\ell_\ell$ span an $\mathfrak{su}(n)$ subalgebra of $\mathfrak{u}(n)$, and the generator $\mathbf{I}$ of $\mathfrak{u}(n)$ spans a $\mathfrak{u}(1)$ subalgebra of $\mathfrak{u}(n)$. Since $\mathbf{I}$ commutes with the $F^\ell_\ell$, it follows that both $\mathfrak{su}(n)$ and $\mathfrak{u}(1)$ are invariant subalgebras (or ideals) of $\mathfrak{u}(n)$. Moreover, the only common element of $\mathfrak{su}(n)$ and $\mathfrak{u}(1)$ is the zero vector. Hence, $\mathfrak{u}(n)\cong\mathfrak{su}(n)\oplus\mathfrak{u}(1)$.

(b) As for the corresponding Lie groups, show that $\mathrm{U}(n)\cong\mathrm{SU}(n)\otimes\mathrm{U}(1)/\mathbb{Z}_n$.

Consider the relation between the Lie groups $\mathrm{SU}(n)\otimes\mathrm{U}(1)$ and $\mathrm{U}(n)$ . In order to determine the corresponding group isomorphism, we first note that any element of $\mathrm{U}(n)$ can be written in the form $e^{i\theta}A$, where $0\leq\theta<2\pi$ and $A$ is a unitary $n\times n$ matrix of unit determinant, and any element of $\mathrm{SU}(n)\times\mathrm{U}(1)$ can be written as an ordered pair, $(A, e^{i\theta})$.

Let us introduce the homomorphism $f$: $\mathrm{SU}(n)\times\mathrm{U}(1)\longrightarrow\mathrm{U}(n)$ that takes $(A, e^{i\theta})\longmapsto e^{i\theta}A$, where $A\in\mathrm{SU}(n)$ and $e^{i\theta}\in\mathrm{U}(1)$. The kernel of the map $f$ consists of all elements of $\mathrm{SU}(n)\times\mathrm{U}(1)$ that are mapped onto the identity element $\mathbf{I}\in\mathrm{U}(n)$. Thus, the elements of the kernel must be of the form $(\mathbf{I}\,e^{-i\theta}\,,\,e^{i\theta})$. In order that $\mathbf{I}\,e^{-i\theta}\in\mathrm{SU}(n)$, we must have

$$\det(\mathbf{I}\,e^{-i\theta}) = e^{-in\theta} = 1\,.$$

It follows that $\theta = 2\pi m/n$ for any integer $m$, and $f(\mathbf{I}\,e^{-2\pi im/n}\,,\,e^{2\pi im/n}) = \mathbf{I}$.

We conclude that[17]

$$\ker f = \left\{(\mathbf{I}\,e^{-2\pi im/n}\,,\,e^{2\pi im/n}),\quad \text{for } m=0,1,2,\ldots,n-1\right\}\cong\mathbb{Z}_n\,. \tag{106}$$

Noting that the image of the map $f$ is given by $\mathrm{im}\,f = \mathrm{U}(n)$, we can use the The Fundamental Theorem of Homomorphisms, which that states that for any homomorphism $f: G\to\mathrm{im}\,f$ with kernel, $\ker f$, we have $\mathrm{im}\,f\cong G/\ker f$. Hence, it then follows that

$$\mathrm{U}(n)\cong\mathrm{SU}(n)\otimes\mathrm{U}(1)/\mathbb{Z}_n\,.$$

[17]Recall the discrete group, $\mathbb{Z}_n = \left\{e^{2\pi im/n},\quad \text{for } m=0,1,2,\ldots,n-1\right\}$. In light of the isomorphism that identifies $(\mathbf{I}\,e^{-2\pi im/n}\,,\,e^{2\pi im/n})\longmapsto e^{2\pi im/n}$, it follows that $\ker f\cong\mathbb{Z}_n$ as indicated in eq. (106).

## APPENDIX: Another simple method for evaluating the determinant of eq. (41)

Consider the following $3 \times 3$ hermitian matrices:

$$A_{nk} = \delta_{nk} - \frac{\xi_n \xi_k}{\xi^2}\,, \qquad B_{nk} = \frac{\xi_n \xi_k}{\xi^2}\,, \qquad C_{nk} = i\epsilon_{nk\ell}\frac{\xi_\ell}{\xi}\,,$$

where the indices $n$ and $k$ run over 1,2,3 and $\xi$ is the magnitude of the vector $\vec{\boldsymbol{\xi}}$. It is easy to verify that these matrices satisfy the following relations:

$$A^2 = A\,, \quad B^2 = B\,, \quad C^2 = C\,, \quad AC = CA = C\,, \quad AB = BA = BC = CB = 0\,.$$

We now define three projection matrices:

$$E_1 = \tfrac{1}{2}(A + C)\,, \qquad E_2 \equiv \tfrac{1}{2}(A - C)\,, \qquad E_3 = B\,, \tag{107}$$

which satisfy:

$$E_i^\dagger = E_i\,, \qquad E_i E_j = \delta_{ij} E_j \ \text{(no sum over } i)\,, \quad \text{and} \quad \sum_{i=1}^{3} E_i = \mathbf{I}\,,$$

where $\mathbf{I}$ is the $3 \times 3$ identity matrix. Using the spectral decomposition of linear algebra, any diagonalizable complex $3 \times 3$ matrix can be written in the form

$$M = \sum_{i=1}^{3} \lambda_i E_i\,, \tag{108}$$

where the $\lambda_i$ are the eigenvalues of $M$ and the $E_i$ are a projection operators that project vectors of $\mathbb{R}^3$ to the one-dimensional linear subspace spanned by the corresponding eigenvector $\vec{\boldsymbol{v}}_i$, respectively. In particular, $E_i \vec{\boldsymbol{v}}_j = \vec{\boldsymbol{v}}_i \delta_{ij}$ (no sum over $i$), which implies [after using eq. (108)] that $M\vec{\boldsymbol{v}}_i = \lambda_j \vec{\boldsymbol{v}}_i$, as expected. The determinant of $M$ is then given by:

$$\det M = \lambda_1 \lambda_2 \lambda_3\,.$$

We now use the above results to evaluate the determinant of eq. (41), which can be rewritten in the following form,

$$c(\vec{\boldsymbol{\xi}}) = \frac{\sin\xi}{\xi} A + B - i\left(\frac{1 - \cos\xi}{\xi}\right) C\,.$$

One can re-express $c(\vec{\boldsymbol{\xi}})$ in terms of the projection operators defined in eq. (107),

$$c(\vec{\boldsymbol{\xi}}) = \left[\frac{\sin\xi}{\xi} - i\left(\frac{1 - \cos\xi}{\xi}\right)\right] E_1 + \left[\frac{\sin\xi}{\xi} + i\left(\frac{1 - \cos\xi}{\xi}\right)\right] E_2 + E_3\,.$$

Comparing with eq. (108), we can read off the eigenvalues of $c(\vec{\boldsymbol{\xi}})$,

$$\lambda_{1,2} = \frac{\sin\xi}{\xi} \mp i\left(\frac{1 - \cos\xi}{\xi}\right)\,, \qquad \lambda_3 = 1\,.$$

Hence,

$$\det c(\vec{\boldsymbol{\xi}}) = \lambda_1 \lambda_2 \lambda_3 = \frac{1}{\xi^2}\left[\sin^2\xi + (1 - \cos\xi)^2\right] = \frac{2(1 - \cos\xi)}{\xi^2}\,,$$

which reproduces the result obtained in eq. (46).