1. This problem concerns the Lie group SO(4) and its Lie algebra $\mathfrak{so}(4)$.

   (a) Work out the Lie algebra $\mathfrak{so}(4)$ and verify that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

The defining representation of the Lie algebra $\mathfrak{so}(n)$ is

$$\mathfrak{so}(n) = \{\mathcal{M} \in \mathfrak{gl}(n, \mathbb{R}) \text{ such that } \mathcal{M}^{\mathsf{T}} = -\mathcal{M}\}\,, \tag{1}$$

where $\mathfrak{gl}(n, \mathbb{R})$ is the set of all real $n \times n$ matrices. Recall that a suitable basis for the defining representation of $\mathfrak{so}(3)$, which consists of all 3×3 real antisymmetric matrices, is $(\mathcal{A}_i)_{jk} = -\epsilon_{ijk}$, where i, j and k can take on the values 1, 2 and 3. To find a suitable basis for the defining representation of $\mathfrak{so}(4)$, one can generalize the $\mathcal{A}_i$ of $\mathfrak{so}(3)$ by choosing

$$(\mathcal{A}_i)_{jk} = \left(\begin{array}{ccc|c} & & & 0 \\ & -\epsilon_{ijk} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \qquad \text{where } i, j, k = 1, 2, 3\,. \tag{2}$$

Since a 4×4 real antisymmetric matrix has six independent parameters, we need to choose three additional linearly-independent antisymmetric matrices to complete the basis for $\mathfrak{so}(4)$. We therefore introduce three antisymmetric matrices $\mathcal{B}_i$ by placing a 1 in one of the non-diagonal elements of the fourth row (and a corresponding -1 required by the antisymmetry property of the matrix), with all other elements zero. That is, a suitable basis for $\mathfrak{so}(4)$ is given by:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{B}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

One can easily verify that the six generators of $\mathfrak{so}(4)$ satisfy the following commutation relations:

$$[\mathcal{A}_i\,, \mathcal{A}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \qquad [\mathcal{B}_i\,, \mathcal{B}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \qquad [\mathcal{A}_i\,, \mathcal{B}_j] = \epsilon_{ijk}\mathcal{B}_k\,. \tag{3}$$

Note that the commutation relations satisfied by the $\mathcal{A}_i$ are precisely those of $\mathfrak{so}(3)$, which is not surprising in light of eq. (2).

The form of the commutators given in eq. (3) is not completely transparent. To understand the implications of eq. (3), it is convenient to define a new set of $\mathfrak{so}(4)$ generators that are real linear combinations of the $\mathcal{A}_i$ and $\mathcal{B}_i$. Thus, we define,

$$X_i \equiv \tfrac{1}{2}(\mathcal{A}_i + \mathcal{B}_i)\,, \qquad Y_i \equiv \tfrac{1}{2}(\mathcal{A}_i - \mathcal{B}_i)\,, \qquad \text{where } i = 1, 2, 3\,. \tag{4}$$

Using eq. (3), it is a simple matter to work out the commutation relations among the X_i and Y_i,

$$[X_i\,,\,X_j] = \epsilon_{ijk}X_k\,, \qquad [Y_i\,,\,Y_j] = \epsilon_{ijk}Y_k\,, \qquad [X_i\,,\,Y_j] = 0\,. \tag{5}$$

Thus, we have succeeding in writing the $\mathfrak{so}(4)$ commutation relations in such a way that the generators $\{X_i\}$ and $\{Y_i\}$ are decoupled. In particular, the $\{X_i\}$ and $\{Y_i\}$ each satisfy $\mathfrak{so}(3)$ commutation relations. Hence, $\mathfrak{so}(4)$ is a direct sum of two independent $\mathfrak{so}(3)$ Lie algebras. That is,[1]

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)\,. \tag{6}$$

(b) What is the universal covering group of SO(4)? What is the center of SO(4)? Identify the adjoint group $\mathrm{Ad}\big(\mathrm{SO}(4)\big)$.

Since the universal covering group of SO(3) is SU(2), we can use eq. (6) to conclude that the universal covering group of SO(4) is SU(2)$\otimes$SU(2).[2] In particular,

$$\mathrm{SO}(4) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)/\mathbb{Z}_2\,. \tag{7}$$

Proof of the isomorphism $\mathbf{SO(4) \cong SU(2) \otimes SU(2)/\mathbb{Z}_2}$

In order to establish eq. (7), it is convenient to consider the set of quaternions $\mathbb{H}$. A quaternion $q \in \mathbb{H}$ is given by $q = a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ and i, j and k satisfy $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The conjugate of q is defined by $q^* = a - bi - cj - dk$, and the magnitude of q is given by $\|q\| = (q^*q)^{1/2} = (qq^*)^{1/2} \in \mathbb{R}$. That is, $\|q\| = (a^2 + b^2 + c^2 + d^2)^{1/2}$. It is easy to prove that $(q_1q_2)^* = q_2^*q_1^*$ (notice the interchange of the order of q_1 and q_2) and $q^{-1} = q^*/||q||^2$ for $q \neq 0$.

There exists a one-to-one and onto map from $\mathbb{H}$ to $\mathbb{R}^4$ that identifies $q = a + bi + cj + dk$ with the four vector (a, b, c, d). There is another one-to-one and onto map from $\mathbb{H}$ to $M_2(\mathbb{C})$ that identifies $q = a + bi + cj + dk$ with the complex 2×2 matrix,[3]

$$q = a + bi + cj + dk \equiv z + wj \longmapsto \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}\,, \qquad \text{where } z = a + bi \text{ and } w = c + di. \tag{8}$$

We shall also introduce the set of unit quaternions, $Q = \big\{q \in \mathbb{H} : ||q|| = 1\big\}$. Note that there exists a one-to-one and onto map from the unit quaternions Q to SU(2) given by eq. (8) where

[1]Since $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ as Lie algebras, we can equally well write $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

[2]Since the Lie group is obtained by exponentiation of the Lie algebra, a direct sum of Lie algebras correspond to a direct product of Lie groups.

[3]We denote $M_2(\mathbb{C})$ as the linear space of all complex 2×2 matrices.

$q \in Q$. This follows from the observation that $1 = \|q\|^2 = |z|^2 + |w|^2$. One can check that given the latter condition, it follows that the complex 2×2 matrix in eq. (8) is unitary with unit determinant.

Consider now a map from $\mathbb{R}^4 \to \mathbb{R}^4$ that is represented by $q \longmapsto q'$, where $q,\, q' \in \mathbb{H}^\star$ (the set of nonzero quaternions) such that $q' \equiv q_1 q q_2^{-1}$ with $q_1,\, q_2 \in Q$. It is easy to check that $\|q'\| = \|q\|$, since

$$\|q'\| = \|q_1 q q_2^{-1}\| = \|q_1 q q_2^*\| = (q_1 q q_2^* q_2 q^* q_1^*)^{1/2} = (qq^*)^{1/2} = \|q\|\,, \tag{9}$$

where we have made use of $\|q_1\| = \|q_2\| = 1$. Thus, the mapping $q \longmapsto q'$ represents a transformation of a four-vector $q \in \mathbb{R}^4$ that leaves the length of q invariant. That is, one can represent the transformation $q \longmapsto q'$ by writing $q' = Rq$, where q and q' are nonzero four-vectors in $\mathbb{R}^4$ and $R \in \mathrm{SO}(4)$. Moreover, since unit quaternions are isomorphic to SU(2), it follows that the mapping $q \longmapsto q_1 q q_2^{-1}$ provides a homomorphism $f : \mathrm{SU}(2) \otimes \mathrm{SU}(2) \to \mathrm{SO}(4)$ corresponding to $(q_1, q_2) \longmapsto R$.

Indeed, all elements of SO(4) can be represented by some mapping of the form $q \longmapsto q'$ [see, e.g., Proposition 8.27 of Ian R. Porteous, *Clifford Algebras and the Classical Groups* (Cambridge University Press, Cambridge, UK, 1995)]. Thus, the image of the homomorphism is given by $\mathrm{im}\, f = \mathrm{SO}(4)$. The kernel of the homomorphism f is the set of pairs of unit quaternions (q_1, q_2) such that $q_1 q q_2^{-1} = q$ for all $q \in \mathbb{H}^\star$. Choosing $q = 1$ yields $q_1 = q_2$, so that $(a, a) \in \ker f$ for $a \in Q$ if and only if $aq = qa$ for all $q \in \mathbb{H}^\star$. We conclude that

$$\ker f = \big\{(1, 1)\,,\,(-1, -1)\big\} \cong \mathbb{Z}_2\,, \tag{10}$$

since 1 and -1 are the only unit quaternions that commute with all nonzero quaternions $q \in \mathbb{H}^\star$. Hence, we can use the fundamental homomorphism theorem of group theory that states that for any homomorphism $f : G \to \mathrm{im}\, f$ with kernel, $\ker f$, we have $\mathrm{im}\, f \cong G/\ker f$. It then follows that

$$\mathrm{SO}(4) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)/\mathbb{Z}_2\,. \tag{11}$$

which confirms eq. (7).

Finally, the adjoint group by definition has a trivial center. Consider the centers of SO(4) and $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$. The center of SO(4) consists of all orthogonal matrices of unit determinant that are multiples of the identity. There are only two such matrices, $\mathbb{1}_{4\times4}$ and $-\mathbb{1}_{4\times4}$, where $\mathbb{1}_{4\times4}$ is the 4×4 identity matrix. Hence,

$$Z(\mathrm{SO}(4)) = \mathbb{Z}_2\,.$$

The center of SU(2) is $\{\mathbb{1}_{2\times2}\,, -\mathbb{1}_{2\times2}\} \cong \mathbb{Z}_2$ so that

$$Z(\mathrm{SU}(2) \otimes \mathrm{SU}(2)) = \mathbb{Z}_2 \otimes \mathbb{Z}_2\,.$$

Thus, the adjoint group of SO(4) can be expressed in a number of equivalent forms,

$$\mathrm{SO}(4)/\mathbb{Z}_2 \cong \mathrm{SO}(3) \otimes \mathrm{SO}(3) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)/\mathbb{Z}_2 \otimes \mathbb{Z}_2\,,$$

where we have made use of the well-known isomorphism, $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$. In particular, $\mathrm{SO}(3) \otimes \mathrm{SO}(3)$ has a trivial center since SO(3) has a trivial center.

(c) Calculate the Killing form of $\mathfrak{so}(4)$ and verify that this Lie algebra is semisimple and compact.

The Cartan-Killing form can be expressed in terms of the Lie algebra structure constants,

$$g_{ab} = f_{ac}^{\;d}\, f_{bd}^{\;c}\,. \tag{12}$$

In this expression, the indices a, b, c and d range over $1, 2, \ldots, 6$, corresponding to the six generators of $\mathfrak{so}(4)$. It is easiest to evaluate g_{ab} in the basis $\{X_i\,,\,Y_j\}$ [cf, eqs. (4) and (5)]. In this basis,

$$f_{ac}^{\;d} = \begin{cases} \epsilon_{ijk}\,, & \text{for } a=i\,, b=j\,, \text{ and } d=k\,, \\ \epsilon_{ijk}\,, & \text{for } a=i+3\,, b=j+3\,, \text{ and } d=k+3\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where i, j and k range over 1, 2 and 3. Plugging into eq. (12) yields

$$g_{ab} = -2\delta_{ab}\,, \tag{13}$$

which indicates that $\mathfrak{so}(4)$ is a semisimple and compact Lie algebra.

(d) Consider the complexification of the corresponding Lie algebras of part (a). Show that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$. Do the conclusions of part (c) still hold? If not, explain how these conclusions are modified?

In part (a), we showed that the most general element of $\mathfrak{so}(4)$ was given by

$$\mathcal{M} = \sum_{i=1}^{3}(a_i\mathcal{A}_i + b_i\mathcal{B}_i) = \begin{pmatrix} 0 & -a_3 & a_2 & -b_1 \\ a_3 & 0 & -a_1 & -b_2 \\ -a_2 & a_1 & 0 & -b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}\,, \qquad \text{for } a_i,\, b_i \in \mathbb{R}. \tag{14}$$

Indeed, $\mathcal{M}$ is a general real antisymmetric 4×4 matrix [cf. eq. (1)].

The complexification of $\mathfrak{so}(4)$ is obtained by taking a_i, $b_i \in \mathbb{C}$ in eq. (14), which yields the most general element of $\mathfrak{so}(4)_{\mathbb{C}}$. Thus, the elements of $\mathfrak{so}(4)_{\mathbb{C}}$ are complex antisymmetric 4×4 matrices. That is,

$$\mathfrak{so}(4)_{\mathbb{C}} = \mathfrak{so}(4,\mathbb{C}) = \{\mathcal{M} \in \mathfrak{gl}(n,\mathbb{C}) \text{ such that } \mathcal{M}^{\mathsf{T}} = -\mathcal{M}\}\,, \tag{15}$$

where $\mathfrak{gl}(n,\mathbb{C})$ is the set of all complex $n\times n$ matrices.

We again define X_i and Y_i as in eq. (4), which satisfy the commutation relations given in eq. (5). Thus, the most general element of $\mathfrak{so}(4)$ can also be written as

$$\sum_{i=1}^{3}(x_iX_i + y_iY_i)\,, \qquad \text{for } x_i,\, y_i \in \mathbb{R}. \tag{16}$$

Since

$$\sum_{i=1}^{3} x_i X_i \in \mathfrak{so}(3) \quad \text{and} \quad \sum_{i=1}^{3} y_i Y_i \in \mathfrak{so}(3)\,, \qquad \text{for } x_i,\, y_i \in \mathbb{R} \tag{17}$$

with $[X_i\,,\,Y_j] = 0$, we concluded in part (a) that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Once again, the complexification of $\mathfrak{so}(4)$ is obtained by taking x_i, $y_i \in \mathbb{C}$ in eq. (16). By a similar argument to the one presented above, it follows that $\mathfrak{so}(3)_{\mathbb{C}} = \mathfrak{so}(3,\mathbb{C})$. Hence, by taking x_i, $y_i \in \mathbb{C}$ in eqs. (16) and (17), we conclude that

$$\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})\,. \tag{18}$$

Since the basis for the Lie algebra did not changed in the process of complexification, it follows that eq. (13) still holds. Thus $g_{ab} = -2\delta_{ab}$, which implies that $\mathfrak{so}(4,\mathbb{C})$ is a semisimple Lie algebra, in light of Cartan's criterion which states that $\det g \neq 0$ if and only if the Lie algebra is semisimple. However, in contrast to the conclusions obtained in part (c), we *cannot* claim that $\mathfrak{so}(4,\mathbb{C})$ is compact. In the class handout entitled *The Cartan-Killing Form*, we proved that if the Killing metric of a semisimple *real* Lie algebra $\mathfrak{g}$ is negative definite then $\mathfrak{g}$ is a compact Lie algebra (which implies that the corresponding Lie group is compact). However, this result does not hold for *complex* semisimple Lie algebras. Indeed, in class we proved a theorem that stated that any compact complex Lie group is abelian.[4] Thus, it follows that any semisimple complex Lie group (which is necessarily nonabelian) must be noncompact.

(e) Using the methods used in part (a), show that $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, where $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is the realification of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. By complexification of this result, show that one recovers the result of part (d).

The defining representation of the Lie algebra $\mathfrak{so}(3,1)$ is

$$\mathfrak{so}(3,1) = \{\mathcal{M} \in \mathfrak{gl}(4,\mathbb{R}) \text{ such that } \mathcal{M}^{\mathsf{T}} = -G\mathcal{M}G^{-1}\}\,, \tag{19}$$

where $G = \mathrm{diag}(+1,+1,+1,-1)$. In this case, the construction given in part (a) is modified. A suitable basis for $\mathfrak{so}(3,1)$ is

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

[4]Consider a complex compact Lie group G and define a map $f(x) = xyx^{-1}y^{-1}$ for some fixed element $y \in G$. Because G is compact, it follows that $f(x)$ is a bounded holomorphic function. Using a well-known theorem of complex analysis, any bounded holomorphic function must be a constant function. Since $f(e) = e$ [where $e \in G$ is the identity element], it follows that $f(x) = e$ for all $x \in G$. That is, $e = xyx^{-1}y^{-1}$, which implies that $xy = yx$ for all $x, y \in G$. Hence, we conclude that a complex compact Lie group must be abelian.

By construction, one can check that $\mathcal{M}^{\mathsf{T}} = -G\mathcal{M}G^{-1}$, where

$$\mathcal{M} = \sum_{i=1}^{3}(a_i\mathcal{A}_i + b_i\mathcal{B}_i) = \begin{pmatrix} 0 & -a_3 & a_2 & b_1 \\ a_3 & 0 & -a_1 & b_2 \\ -a_2 & a_1 & 0 & b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}, \qquad \text{for } a_i,\, b_i \in \mathbb{R}. \tag{20}$$

One can easily verify that the six generators of $\mathfrak{so}(3,1)$ satisfy the following commutation relations:

$$[\mathcal{A}_i\,,\,\mathcal{A}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \qquad [\mathcal{B}_i\,,\,\mathcal{B}_j] = -\epsilon_{ijk}\mathcal{A}_k\,, \qquad [\mathcal{A}_i\,,\,\mathcal{B}_j] = \epsilon_{ijk}\mathcal{B}_k\,. \tag{21}$$

The most general element of $\mathfrak{sl}(2,\mathbb{C})$ is $\sum_i c_i e_i$, where $c_i \in \mathbb{C}$, and the generators e_i satisfy the commutation relations,

$$[e_i, e_j] = \epsilon_{ijk}e_k\,. \tag{22}$$

The realification of $\mathfrak{sl}(2,\mathbb{C})$ is obtained by choosing a basis $\{e_1, e_2, e_3, ie_1, ie_2, ie_3\}$ and considering *real* linear combinations of the six generators. It is convenient to denote $f_i \equiv ie_i$ for $i = 1, 2, 3$. Then, the most general element of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is given by

$$\sum_{i=1}^{3}(c_i e_i + d_i f_i)\,, \qquad \text{for } c_i,\, d_i \in \mathbb{R}. \tag{23}$$

Using eq. (22) and $f_i = ie_i$, it immediately follows that

$$[e_i\,,\,e_j] = \epsilon_{ijk}e_k\,, \qquad [f_i\,,\,f_j] = -\epsilon_{ijk}e_k\,, \qquad [e_i\,,\,f_j] = \epsilon_{ijk}f_k\,. \tag{24}$$

Comparing eqs. (21) and (24), one can conclude that

$$\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\,. \tag{25}$$

The final step is to complexify eq. (25). First, consider the complexification of $\mathfrak{so}(3,1)$. The six generators of $\mathfrak{so}(3,1)_{\mathbb{C}}$ satisfy the commutation relations given in eq. (21). We shall choose a new basis of six generators, which consists of $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_2, \widetilde{\mathcal{B}}_3\}$, where

$$\widetilde{\mathcal{B}}_j \equiv i\mathcal{B}_j\,. \tag{26}$$

Note that the multiplication by i is permissible, since $\mathfrak{so}(3,1)_{\mathbb{C}}$ is a complex Lie algebra. Then, the commutation relations of the new $\mathfrak{so}(3,1)_{\mathbb{C}}$ generators are given by

$$[\mathcal{A}_i\,,\,\mathcal{A}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \qquad [\widetilde{\mathcal{B}}_i\,,\,\widetilde{\mathcal{B}}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \qquad [\mathcal{A}_i\,,\,\widetilde{\mathcal{B}}_j] = \epsilon_{ijk}\widetilde{\mathcal{B}}_k\,. \tag{27}$$

Comparing eqs. (27) and (3), we conclude that

$$\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{so}(4,\mathbb{C})\,. \tag{28}$$

Our remaining task is to complexify $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$. The complexification of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is obtained by taking c_i, $d_i \in \mathbb{C}$ in eq. (23). The commutation relations of the generators of $[\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}]_{\mathbb{C}}$

are still given by eq. (24). As previously noted, since $[\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}]_{\mathbb{C}}$ is a complex Lie algebra, one is permitted to define a new basis by taking *complex* linear combinations of the generators. In analogy with eq. (4), we define

$$X_j \equiv \tfrac{1}{2}(e_j + if_j)\,, \qquad\qquad Y_j \equiv \tfrac{1}{2}(e_j - if_j)\,, \qquad \text{where } j = 1, 2, 3\,. \tag{29}$$

Using eq. (24), it is a simple matter to work out the commutation relations among the X_i and Y_i,

$$[X_i\,,\,X_j] = \epsilon_{ijk} X_k\,, \qquad [Y_i\,,\,Y_j] = \epsilon_{ijk} Y_k\,, \qquad [X_i\,,\,Y_j] = 0\,. \tag{30}$$

Thus, we have succeeding in writing the $[\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}]_{\mathbb{C}}$ commutation relations in such a way that the generators $\{X_i\}$ and $\{Y_i\}$ are decoupled, and the $\{X_i\}$ and $\{Y_i\}$ each satisfy $\mathfrak{so}(3,\mathbb{C})$ commutation relations. In particular,

$$\sum_{i=1}^{3} x_i X_i \in \mathfrak{so}(3,\mathbb{C}) \quad \text{and} \quad \sum_{i=1}^{3} y_i Y_i \in \mathfrak{so}(3,\mathbb{C})\,, \qquad \text{for } x_i,\, y_i \in \mathbb{C}, \tag{31}$$

with $[X_i\,,\,Y_j] = 0$. Hence, $\mathfrak{so}(4,\mathbb{C})$ is a direct sum of two independent $\mathfrak{so}(3,\mathbb{C})$ Lie algebras. That is,[5]

$$\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})\,, \tag{32}$$

thereby reproducing the result of eq. (18).

<u>*BONUS MATERIAL*</u>

Define the following family of real Lie algebras that are subalgebras of the real Lie algebra of $n \times n$ matrices:[6]

$$\mathfrak{so}(r,s,G) = \bigl\{\mathcal{M} \in \mathfrak{gl}(n,\mathbb{R}) \text{ such that } \mathcal{M}^{\mathsf{T}} = -G\mathcal{M}G^{-1}\bigr\}\,, \tag{33}$$

where G is a real symmetric invertible $n \times n$ matrix that possesses r positive eigenvalues and s negative eigenvalues with $n = r + s$. We now make use of Sylvester's Theorem (see Corollary 1 of Appendix A of the class handout entitled *The Cartan-Killing Form*), which states that an invertible real matrix R exists such that

$$R^{\mathsf{T}} G R = D(r,s) \equiv \text{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s})\,. \tag{34}$$

We can also introduce the real Lie algebra $\mathfrak{so}(r,s)$ which is defined by

$$\mathfrak{so}(r,s) = \bigl\{\mathcal{N} \in \mathfrak{gl}(n,\mathbb{R}) \text{ such that } \mathcal{N}^{\mathsf{T}} = -D\mathcal{N}D^{-1} \text{ and } D = D(r,s)\bigr\}\,. \tag{35}$$

The following isomorphism of Lie algebras is noteworthy:

$$\mathfrak{so}(r,s,G) \cong \mathfrak{so}(r,s)\,, \tag{36}$$

[5]Since $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C})$ as Lie algebras, we can equally well write $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

[6]One can check that if $\mathcal{M}_1,\ \mathcal{M}_2 \in \mathfrak{so}(r,s,G)$ then $[\mathcal{M}_1,\ \mathcal{M}_2] \in \mathfrak{so}(r,s,G)$, thereby confirming that $\mathfrak{so}(r,s,G)$ is a Lie algebra.

where G is a real symmetric invertible $n \times n$ matrix that possesses r positive eigenvalues and s negative eigenvalues with $n = r + s$. To exhibit this isomorphism, consider the function

$$\mathcal{N} = f(\mathcal{M}) = R^{-1}\mathcal{M}R\,, \quad \text{where } \mathcal{M} \in \mathfrak{so}(r,s,G) \text{ and } R^{\mathsf{T}}GR = D(r,s)\,. \tag{37}$$

In light of eq. (33), $\mathcal{M}^{\mathsf{T}} = -G\mathcal{M}G^{-1}$. Using eq. (37) to write $\mathcal{M} = R\mathcal{N}R^{-1}$, it then follows that

$$(R\mathcal{N}R^{-1})^{\mathsf{T}} = -GR\mathcal{N}R^{-1}G^{-1}\,. \tag{38}$$

Multiplying eq. (38) by R^{T} on the left and by $R^{\mathsf{T}-1}$ on the right and simplifying the resulting equation, we end up with

$$\mathcal{N}^{\mathsf{T}} = -(R^{\mathsf{T}}GR)\mathcal{N}(R^{\mathsf{T}}GR)^{-1}\,. \tag{39}$$

Using eq. (34), it follows that

$$\mathcal{N}^{\mathsf{T}} = -D\mathcal{N}D^{-1}\,. \tag{40}$$

That is, $\mathcal{N} \in \mathfrak{so}(r,s)$ [cf. eq. (35)]. In particular, the function $f(\mathcal{M})$ defined in eq. (37) is a mapping from $\mathfrak{so}(r,s,G)$ to $\mathfrak{so}(r,s)$. It is easy to check that this mapping is bijective (one-to-one and onto), which establishes the isomorphism announced in eq. (36).

Consider now the complexification of $\mathfrak{so}(r,s,G)$, which is obtained by replacing $\mathbb{R}$ with $\mathbb{C}$ in eq. (33),

$$\mathfrak{so}(r,s,G)_{\mathbb{C}} = \big\{\mathcal{M} \in \mathfrak{gl}(n,\mathbb{C}) \text{ such that } \mathcal{M}^{\mathsf{T}} = -G\mathcal{M}G^{-1}\big\}\,, \tag{41}$$

where G is a complex symmetric invertible $n \times n$ matrix. In this case, we can employ the extension of Sylvester's Theorem given in Appendix B of the class handout entitled *The Cartan-Killing Form*, which states that an invertible complex matrix S exists such that

$$S^{\mathsf{T}}GS = \mathbb{1}_{n\times n}\,, \tag{42}$$

where $\mathbb{1}_{n\times n}$ is the $n \times n$ identity matrix. We can make use of this result to establish the Lie algebra isomorphism,

$$\mathfrak{so}(r,s,G)_{\mathbb{C}} \cong \mathfrak{so}(r+s,\mathbb{C})\,, \tag{43}$$

where

$$\mathfrak{so}(n,\mathbb{C}) = \big\{\mathcal{N} \in \mathfrak{gl}(n,\mathbb{C}) \text{ such that } \mathcal{N}^{\mathsf{T}} = -\mathcal{N}\big\}\,. \tag{44}$$

Consider the function

$$\mathcal{N} = f(\mathcal{M}) = S^{-1}\mathcal{M}S\,, \quad \text{where } \mathcal{M} \in \mathfrak{so}(r,s,G)_{\mathbb{C}} \text{ and } S^{\mathsf{T}}GS = \mathbb{1}_{n\times n} \text{ (with } n = r+s)\,. \tag{45}$$

Using eq. (45) to write $\mathcal{M} = S\mathcal{N}S^{-1}$, it then follows from eq. (41) that

$$(S\mathcal{N}S^{-1})^{\mathsf{T}} = -GS\mathcal{N}S^{-1}G^{-1}\,. \tag{46}$$

Multiplying eq. (46) by S^{T} on the left and by $S^{\mathsf{T}-1}$ on the right and simplifying the resulting equation, we end up with

$$\mathcal{N}^{\mathsf{T}} = -(S^{\mathsf{T}}GS)\mathcal{N}(S^{\mathsf{T}}GS)^{-1} = -\mathcal{N}\,, \tag{47}$$

after employing eq. (42). In light of eq. (44), it follows that $\mathcal{N} \in \mathfrak{so}(n,\mathbb{C})$, where $n = r+s$. In particular, the function $f(\mathcal{M})$ defined in eq. (45) is a mapping from $\mathfrak{so}(r,s,G)_{\mathbb{C}}$ to $\mathfrak{so}(r+s,\mathbb{C})$.

It is easy to check that this mapping is bijective (one-to-one and onto), which establishes the isomorphism announced in eq. (43).

Finally, by combining eqs. (36) and (43), we deduce the following isomorphism of Lie algebras,

$$\mathfrak{so}(r,s)_{\mathbb{C}} \cong \mathfrak{so}(r+s,\mathbb{C})\,, \tag{48}$$

where $\mathfrak{so}(r,s)_{\mathbb{C}}$ is the complexification of $\mathfrak{so}(r,s)$. Setting $r=3$ and $s=1$, we recover the result previously obtained in eq. (28) by another method.

2. A Lie algebra $\mathfrak{g}$ is defined by the commutation relations of the generators,

$$[e_a, e_b] = f_{ab}^{c} e_c\,.$$

Consider the finite-dimensional matrix representations of the e_a. We shall denote the corresponding generators in the adjoint representation by F_a and in an arbitrary irreducible representation R by R_a. The dimension of the adjoint representation, d, is equal to the dimension of the Lie algebra $\mathfrak{g}$, while the dimension of R will be denoted by d_R.

(a) Show that the Cartan-Killing metric g_{ab} can be written as $g_{ab} = \mathrm{Tr}(F_aF_b)$.

The Cartan-Killing metric can be expressed in terms of the structure constants as follows,

$$g_{ij} = f_{i\ell}^{k} f_{jk}^{\ell}\,.$$

On the other hand, matrix elements of the adjoint representation are given by:

$$(F_i)^j{}_k = f_{ik}^{j}\,,$$

where j labels the rows and k labels the columns of the matrices F_i. Therefore,

$$\mathrm{Tr}(F_iF_j) = (F_i)^k{}_\ell (F_j)^\ell{}_k = f_{i\ell}^{k} f_{jk}^{\ell} = g_{ij}\,.$$

(b) If $\mathfrak{g}$ is a simple real compact Lie algebra, prove that for any irreducible representation R,

$$\mathrm{Tr}(R_aR_b) = c_R g_{ab}\,,$$

where c_R is called the *index* of the irreducible representation R.

Consider a d-dimensional Lie algebra $\mathfrak{g}$, whose generators are represented by the matrices R_a. These matrices satisfy the Lie algebra commutation relations,

$$[R_a, R_b] = f_{ab}^{c} R_c\,, \qquad \text{where } a,b,c = 1,2,\ldots,d\,. \tag{49}$$

We first note the following identity:

$$\mathrm{Tr}\big\{[R_a,R_b]R_c\big\} = \mathrm{Tr}\big\{R_a[R_b,R_c]\big\}\,. \tag{50}$$

The proof of eq. (50) is straightforward:

$$\begin{aligned}\mathrm{Tr}\big\{[R_a, R_b]R_c\big\} &= \mathrm{Tr}\big\{(R_aR_b - R_bR_a)R_c\big\} = \mathrm{Tr}(R_aR_bR_c) - \mathrm{Tr}(R_bR_aR_c) \\ &= \mathrm{Tr}(R_aR_bR_c) - \mathrm{Tr}(R_aR_cR_b) = \mathrm{Tr}\big\{R_a(R_bR_c - R_cR_b)\big\} = \mathrm{Tr}\big\{R_a[R_b, R_c]\big\}\,,\end{aligned}$$

after using the cyclic properties of the trace. Making use of eq. (49) in eq. (50) yields:

$$f_{ab}^{d}\,\mathrm{Tr}(R_dR_c) = f_{bc}^{d}\,\mathrm{Tr}(R_aR_d)\,. \tag{51}$$

To make further progress, recall that $f_{abc} \equiv g_{ad}f_{bc}^{d}$ is totally antisymmetric under the interchange of any pair of indices a, b and c. It follows that

$$f_{bc}^{d} = g^{ad}f_{abc}\,, \tag{52}$$

where g^{ad} is the inverse Cartan metric tensor. It is convenient to multiply both sides of eq. (51) by g^{ea} to obtain:

$$g^{ea}f_{ab}^{d}\,\mathrm{Tr}(R_dR_c) = g^{ea}f_{bc}^{d}\,\mathrm{Tr}(R_aR_d)\,. \tag{53}$$

Using eq. (52) and the antisymmetry properties of f_{abh},

$$g^{ea}f_{ab}^{d} = g^{ea}g^{hd}f_{hab} = g^{ea}g^{hd}f_{abh} = g^{hd}f_{bh}^{e}\,.$$

Inserting this result into eq. (53) yields

$$g^{hd}f_{bh}^{e}\,\mathrm{Tr}(R_dR_c) = g^{ea}f_{bc}^{d}\,\mathrm{Tr}(R_aR_d)\,. \tag{54}$$

Consider the $d \times d$ matrix whose matrix elements are $A^h{}_c \equiv g^{hd}\,\mathrm{Tr}(R_dR_c)$. We can then rewrite eq. (54) in the following form:

$$f_{bh}^{e}A^h{}_c = f_{bc}^{d}A^e{}_d\,. \tag{55}$$

We recognize $f_{bh}^{e} = (F_b)^e{}_h$ and $f_{bc}^{d} = (F_b)^d{}_c$. Hence, eq. (55) is equivalent to the ec component of the matrix equation,

$$F_bA = AF_b\,,$$

for all $b = 1, 2, \ldots, d$.

We proved in class that the adjoint representation of a simple Lie algebra (whose generators are represented by the matrices F_b) is irreducible. Applying Schur's second lemma to representations of Lie algebras,[7] any matrix that commutes with all the F_b must be a multiple of the identity. Hence, $A = c\mathbf{I}$ or equivalently.

$$g^{ed}\,\mathrm{Tr}(R_dR_c) = c_R\delta_c^e\,,$$

[7] A review of the proof given in class of Schur's lemmas (which were applied to group representations) reveals that it also applies to representations of Lie algebras. Indeed, for any algebraic structure $\mathscr{A}$, Schur's second lemma states that if there exists a matrix M such that $D(\mathcal{A})M = MD(\mathcal{A})$ for *all* $\mathcal{A} \in \mathscr{A}$, where $D(\mathcal{A})$ is an n-dimensional irreducible matrix representation of $\mathcal{A}$ (over a complex representation space $\mathbb{C}^n$), then it follows that M must be a multiple of the identity matrix. In particular, any element of a Lie algebra $\mathscr{A}$ can be expressed as some linear combination of the generators $\mathcal{A}_a$ (which serve as a basis for the Lie algebra). Consequently, if $D(\mathcal{A}_a)M = MD(\mathcal{A}_a)$ for all $a = 1, 2, \ldots, d$, then it follows that $D(\mathcal{A})M = MD(\mathcal{A})$ for all $\mathcal{A} \in \mathscr{A}$, and Schur's second lemma applies.

where c_R is some complex constant. Using $g^{ed}g_{eh} = \delta^d_h$, it immediately follows that

$$\mathrm{Tr}(R_h R_c) = c_R g_{hc}\,, \tag{56}$$

which is the desired result.

(c) The quadratic Casimir operator is defined as $C_2 \equiv g^{ab}e_a e_b$ where g^{ab} is the inverse of g_{ab}. Recall that C_2 commutes with all elements of the Lie algebra. Hence, by Schur's lemma, C_2 must be a multiple of the identity operator. Let us write $C_2 = C_2(R)\mathbf{I}$, where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix and $C_2(R)$ is the eigenvalue of the Casimir operator in the irreducible representation R. As noted above, d is the dimension of the Lie algebra $\mathfrak{g}$. Show that $C_2(R)$ is related to the index c_R by

$$C_2(R) = \frac{dc_R}{d_R}\,.$$

Check this formula in the case that R is the adjoint representation.

By definition,

$$C_2(R)\mathbf{I} = g^{ab}R_a R_b\,, \tag{57}$$

where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix, d_R is the dimension of the representation R, and $a, b = 1, 2, \ldots, d$. Taking the trace of eq. (57) and using eq. (56), it follows that:

$$d_R C_2(R) = g^{ab}\mathrm{Tr}(R_a R_b) = c_R g^{ab} g_{ab} = c_R d\,,$$

since $g^{ab}g_{ab} = \delta^a_a = d$. Hence, solving for $C_2(R)$, one obtains:

$$C_2(R) = \frac{dc_R}{d_R}\,. \tag{58}$$

For the adjoint representation (usually denoted by $R = A$), we have $d_A = d$. Moreover, the adjoint representation generators are $(R_a)^b{}_c = f^b_{ac}$, as shown in class. Hence,

$$\mathrm{Tr}(R_a R_d) = (R_a)^b{}_c (R_d)^c{}_b = f^b_{ac} f^c_{db} = g_{ad}\,,$$

where we used the definition of the Cartan metric tensor at the last step. Comparing this result with that of eq. (56) yields $c_A = 1$. Hence, eq. (58) implies that $C_2(A) = 1$ in agreement with the theorem proved in class.

(d) Compute the index of an arbitrary irreducible representation of $\mathfrak{su}(2)$.

For $\mathfrak{su}(2)$, the irreducible representations are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$. The quadratic Casimir operator is proportional to $J_1^2 + J_2^2 + J_3^2$, where $[J_i, J_j] = i\epsilon_{ijk}J_k$ in the physicist's convention. Since the eigenvalue of $J_1^2 + J_2^2 + J_3^2$ is $j(j+1)$, we shall adjust the overall normalization of the Casimir operator so that $C_2(A) = 1$. Given that the adjoint representation of $\mathfrak{su}(2)$ corresponds to $j = 1$, it follows that:

$$C_2(j) = \tfrac{1}{2}j(j+1)\,.$$

We now use eq. (58) to obtain the index of an irreducible representation of $\mathfrak{su}(2)$. Using $d_R = 2j+1$ for the irreducible representation labeled by j, it follows that the index c_R is

$$c(j) = \tfrac{1}{6}j(j+1)(2j+1)\,.$$

In the defining representation, $j=\frac{1}{2}$, and we find $c_F \equiv c(\frac{1}{2}) = \frac{1}{4}$. In the adjoint representation, $j=1$ and we find that $c_A \equiv c(1) = 1$ as expected from part (b).

(e) Compute the index of the defining representation of $\mathfrak{su}(3)$ and generalize this result to $\mathfrak{su}(n)$.

First, consider the Lie algebra $\mathfrak{su}(3)$. We choose the generators in the defining representation to be the Gell-Mann matrices, $\frac{1}{2}\lambda_a$. Following the mathematician's conventions, we define $T_a \equiv -\frac{1}{2}i\lambda_a$ so that

$$[T_a, T_b] = f_{abc}T_c\,,$$

where the f_{abc} are the totally antisymmetric structure constants in the convention where the T_a satisfy

$$\mathrm{Tr}(T_aT_b) = -\tfrac{1}{4}\mathrm{Tr}(\lambda_a\lambda_b) = -\tfrac{1}{2}\delta_{ab}\,, \tag{59}$$

using the explicit form for the Gell-Mann matrices given in the class handout entitled *Properties of the Gell-Mann matrices.* In this basis choice,

$$g_{ab} = f_{ad}^{\,c} f_{bd}^{\,c} = -3\delta_{ab}\,,$$

using the explicit form for the $\mathfrak{su}(3)$ structure constants listed in the class handout on SU(3). The index of the defining representation, usually denoted by c_F (since physicists also refer to this representation as the fundamental representation), can be obtained from eq. (56),

$$\mathrm{Tr}(T_aT_b) = c_F(-3\delta_{ab})\,.$$

Using eq. (59) to compute the trace, we end up with $c_F = \frac{1}{6}$.

To generalize these results to $\mathfrak{su}(n)$, we shall make use of the construction of the $\mathfrak{su}(n)$ Lie algebra given in the class handout entitled *Properties of the Gell-Mann matrices.* There, we defined traceless $n\times n$ matrices,

$$(F^a_b)_{cd} = \delta_{bc}\delta^a_d - \frac{1}{n}\delta^a_b\delta_{cd}\,,$$

which satisfy the commutation relations,

$$[F^a_b, F^c_d] = \delta^a_d F^c_b - \delta^c_b F^a_d\,. \tag{60}$$

The generalized (hermitian) Gell-Mann matrices are:

$$\lambda_1 = F^1_2 + F^2_1 = \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \lambda_2 = i(F^1_2 - F^2_1) = \left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & 0 \end{array}\right),$$

$$\lambda_3 = F^1_1 - F^2_2 = \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \text{etc.} \tag{61}$$

where the Pauli matrices occupy the upper left 2×2 block of the $n \times n$ matrix generators (with all other elements zero). In the mathematician's convention, we define $T_a = -\frac{1}{2} i\lambda_a$ and $[T_a, T_b] = f_{abc} T_c$, where the f_{abc} are totally antisymmetric and $\mathrm{Tr}(T_a T_b) \propto \delta_{ab}$. To compute the constant of proportionality, one can check for example that

$$\mathrm{Tr}(T_3 T_3) = -\tfrac{1}{4} \mathrm{Tr}(\lambda_3 \lambda_3) = -\tfrac{1}{2}\,,$$

using eq. (61). Clearly, the constant of proportionality does not depend on the choice of a and b. Hence, it follows that the generators of $\mathfrak{su}(n)$ in the defining representation satisfy

$$\mathrm{Tr}(T_a T_b) = -\tfrac{1}{2} \delta_{ab}\,. \tag{62}$$

Next, we evaluate the Cartan metric tensor, which is given by:

$$g_{ab} = f_{ad}^{c} f_{bc}^{d}\,. \tag{63}$$

In the convention where the generators satisfy $\mathrm{Tr}(T_a T_b) \propto \delta_{ab}$, the Cartan metric tensor also satisfies $g_{ab} \propto \delta_{ab}$, in light of eq. (56). To determine the proportionality constant, consider

$$[T_3, T_c] = f_{3cd} T_d\,.$$

We can evaluate $g_{33} = f_{3dc} f_{3cd}$ by examining eq. (60). In particular,

$$\begin{aligned}
&[T_3, F_1^2] = F_1^2\,, \qquad [T_3, F_2^1] = -F_2^1\,, \qquad [T_3, F_1^a] = \tfrac{1}{2} F_1^a\,, \qquad [T_3, F_a^1] = -\tfrac{1}{2} F_a^1\,, \\
&[T_3, F_2^a] = -\tfrac{1}{2} F_1^a\,, \quad [T_3, F_a^2] = \tfrac{1}{2} F^1{}_a\,, \qquad [T_3, F_b^a] = [T_3, F_a^b] = 0\,,
\end{aligned} \tag{64}$$

for $a \neq b$ and $a, b = 3, 4, \ldots, n$. Note that the non-diagonal generators T_c of the form $F_b^a + F_a^b$ and $i(F_b^a - F_a^b)$ for $a < b$ with $a = 1$ or $a = 2$ are the only generators that do not commute with T_3. Eq. (64) provides the necessary information to evaluate g_{33},

$$g_{33} = (+1)(-1) + (n-1)\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) = -n\,.$$

where the first term on the right-hand side derives from $f_{312} f_{321}$, whereas the remaining terms derive from the remaining combination of non-zero structure constants. That is,

$$g_{ab} = f_{ad}^{c} f_{bc}^{d} = -n \delta_{ab}\,.$$

The index of the defining representation can be obtained from eq. (56),

$$\mathrm{Tr}(T_a T_b) = c_F(-n \delta_{ab})\,.$$

Using eq. (62) to compute the trace, we end up with

$$c_F = \frac{1}{2n}\,. \tag{65}$$

One sees that this general result is consistent with the corresponding results of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ previously obtained.

Remarks:

Using eqs. (58) and (65), one can compute the eigenvalue of the quadratic Casimir operator in the defining representation of $\mathfrak{su}(n)$. In particular, since $d = n^2-1$, $d_F = n$ and $c_F = 1/(2n)$, it follows that:

$$C_2(F) = \frac{n^2-1}{2n^2}\,.$$

Moreover, the Casimir operator in the defining representation of $\mathfrak{su}(n)$ is given by

$$C_2(A) = 1\,,$$

according to the theorem proved in class. However, note that the Casimir operator of $\mathfrak{su}(n)$ is defined in an arbitrary irreducible representation R by

$$C_2 = g^{ab}R_aR_b = -\frac{1}{n}\sum_{a=1}^{n^2-1} R_aR_a\,, \tag{66}$$

where we have used eq. (63) [recall that g^{ab} is the inverse of g_{ab}]. In the physics literature, in the case of $\mathfrak{su}(n)$ one typically defines C_2 by omitting the overall factor of $1/n$ in eq. (66). Consequently, $C_2(R)$ is a factor of n larger than indicated above, in which case

$$C_2(F) = \frac{n^2-1}{2n}\,, \qquad\qquad C_a(A) = n\,.$$

Additional details on the Casimir operator and index of an irreducible representation of a simple Lie algebra can be found in the class handout entitled, *The eigenvalues of the quadratic Casimir operator and second-order indices of a simple Lie algebra.*

3. Various subalgebras of $\mathfrak{su}(3)$ may be identified with specific subsets of the $\mathfrak{su}(3)$ generators.

(a) Show that the Gell-Mann matrices λ_1, λ_2, and λ_3 generate an $\mathfrak{su}(2)$ subalgebra.

Consider the commutation relations satisfied by λ_1, λ_2 and λ_3,

$$[\lambda_a, \lambda_b] = 2i\epsilon_{abc}\lambda_c\,, \qquad \text{for } a,b,c = 1,2,3\,.$$

If we define $T_a \equiv -\frac{1}{2}i\lambda_a$, then the resulting commutation relations,

$$[T_a, T_b] = \epsilon_{ijk}T_c\,, \qquad \text{for } a,b,c = 1,2,3\,, \tag{67}$$

correspond to an $\mathfrak{su}(2)$ Lie algebra, which is a subalgebra of the $\mathfrak{su}(3)$ Lie algebra.

(b) Show that the Gell-Mann matrices λ_2, λ_5, and λ_7 generate an $\mathfrak{so}(3)$ subalgebra. Why do you think I called this an $\mathfrak{so}(3)$ subalgebra rather than an $\mathfrak{su}(2)$ subalgebra?

Consider the commutation relations, $[\lambda_2, \lambda_5] = i\lambda_7$, and cyclic permutations thereof. It follows that $\{-i\lambda_2, -i\lambda_5, -i\lambda_7\}$ satisfy the same $\mathfrak{su}(2)$ commutation relations as the T_a of eq. (67). Indeed, the matrix forms of $\{-i\lambda_2, -i\lambda_5, -i\lambda_7\}$ are:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{68}$$

which are of the form $(\mathcal{A}_a)_{bc} = -\epsilon_{abc}$.

The matrices given in eq. (68) constitute the adjoint representation of the generators of $\mathfrak{su}(2)$. When exponentiated, these matrices generate the Lie group SO(3), since SO(3) is the adjoint group of SU(2). Hence, we say that $\{-i\lambda_2, -i\lambda_5, -i\lambda_7\}$ generate an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{su}(3)$.

(c) Decompose (if necessary) the three-dimensional irreducible representation of $\mathfrak{su}(3)$ into representations that are irreducible under the subalgebras of parts (a) and (b).

If we decompose the three-dimensional irreducible representation of $\mathfrak{su}(3)$ denoted henceforth by **3**, with respect to the $\mathfrak{su}(2)$ subalgebra that is generated by $\{-i\lambda_1,\ -i\lambda_2,\ -i\lambda_3\}$, then it is easy to determine from the weight diagram shown in Fig. 1 the components of the weight vectors of the **3** corresponding to the eigenvalues of $T_3 \equiv \frac{1}{2}\lambda_3$.

In particular, the **3** of $\mathfrak{su}(3)$ contains a doublet $\binom{u}{d}$ with $T_3 = \pm\frac{1}{2}$ and a singlet s with $T_3 = 0$. That is, with respect to the $\mathfrak{su}(2)$ subalgebra generated by $\{-i\lambda_1,\ -i\lambda_2,\ -i\lambda_3\}$, the **3** of $\mathfrak{su}(3)$ decomposes as

$$\mathbf{3} \longrightarrow \mathbf{2} \oplus \mathbf{1}\,.$$

This is an example of a *branching rule.*

The decomposition of the **3** of $\mathfrak{su}(3)$ with respect to the $\mathfrak{su}(2)$ subalgebra generated by $\{-i\lambda_2,\ -i\lambda_5,\ -i\lambda_7\}$ is obtained as follows. In part (b), we noted that the explicit form for the

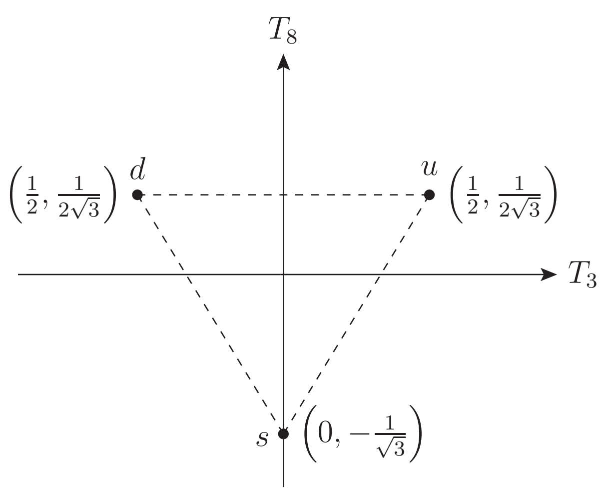

Figure 1: The weight diagram of the three-dimensional defining representation, **3**, of $\mathfrak{su}(3)$.

matrices $\{-i\lambda_2,\, -i\lambda_5,\, -i\lambda_7\}$ are given by $(\mathcal{A}_a)_{bc} = -\epsilon_{abc}$, which is the adjoint representation for the generators of $\mathfrak{su}(2)$. The latter is a three-dimensional irreducible representation of $\mathfrak{su}(2)$. Hence, in this case, the corresponding branching rule is

$$\mathbf{3} \longrightarrow \mathbf{3}\,. \tag{69}$$

Since the adjoint group of SU(2) is SO(3), it is appropriate to consider the branching rule as describing the embedding of an $\mathfrak{so}(3)$ subalgebra within the Lie algebra $\mathfrak{su}(3)$.

4. Consider the simple Lie algebra $\mathfrak{g}$ generated by the ten 4×4 matrices: $\sigma_a\otimes\mathbf{I}$, $\sigma_a\otimes\tau_1$, $\sigma_a\otimes\tau_3$ and $\mathbf{I}\otimes\tau_2$, where $(\mathbf{I},\sigma_a)$ and $(\mathbf{I},\tau_a)$ are the 2×2 identity and Pauli matrices in orthogonal spaces. For example, since $\tau_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, we obtain in block matrix form:

$$\sigma_a\otimes\tau_3 = \left(\begin{array}{c|c} \sigma_a & \mathbf{0} \\ \hline \mathbf{0} & -\sigma_a \end{array}\right)\,, \qquad (a=1,2,3)\,,$$

where $\mathbf{0}$ is the 2×2 zero matrix. The remaining seven matrices can be likewise obtained. Take $H_1 = \sigma_3\otimes\mathbf{I}$ and $H_2 = \sigma_3\otimes\tau_3$ as the generators of the Cartan subalgebra. Note that if A, B, C, and D are 2×2 matrices, then $(A\otimes B)(C\otimes D) = AC\otimes BD$.

(a) Find the roots. Normalize the roots such that the shortest root vector has length 1. What is the rank of $\mathfrak{g}$?

First, we write out the ten generators explicitly in block matrix form:

$$A_a \equiv \sigma_a\otimes\tau_1 = \left(\begin{array}{c|c} \mathbf{0} & \sigma_a \\ \hline \sigma_a & \mathbf{0} \end{array}\right)\,, \qquad (a=1,2,3)\,,$$

$$B_a \equiv \sigma_a\otimes\tau_3 = \left(\begin{array}{c|c} \sigma_a & \mathbf{0} \\ \hline \mathbf{0} & -\sigma_a \end{array}\right)\,, \qquad (a=1,2,3)\,,$$

$$C_a \equiv \sigma_a\otimes\mathbf{I} = \left(\begin{array}{c|c} \sigma_a & \mathbf{0} \\ \hline \mathbf{0} & \sigma_a \end{array}\right)\,, \qquad (a=1,2,3)\,,$$

$$D \equiv \mathbf{I}\otimes\tau_2 = \left(\begin{array}{c|c} \mathbf{0} & -i\mathbf{I} \\ \hline i\mathbf{I} & \mathbf{0} \end{array}\right)\,. \tag{70}$$

To check that these generators actually generate a Lie algebra, we work out all the commutation relations:

$$\begin{aligned}
&[A_a, A_b] = 2i\epsilon_{abc}C_c\,, && [B_a, B_b] = 2i\epsilon_{abc}C_c\,, && [C_a, C_b] = 2i\epsilon_{abc}C_c\,, \\
&[A_a, B_b] = -2i\delta_{ab}D\,, && [A_a, C_b] = 2i\epsilon_{abc}A_c\,, && [B_a, C_b] = 2i\epsilon_{abc}B_c\,, \\
&[A_a, D] = 2iB_a\,, && [B_a, D] = -2iA_a\,, && [C_a, D] = 0\,,
\end{aligned} \tag{71}$$

where we have used $\sigma_a\sigma_b = \mathbf{I}\delta_{ab} + i\epsilon_{abc}\sigma_c$. For example,

$$
\begin{aligned}
[A_a, B_b] = A_a B_b - B_b A_a &= \left(\begin{array}{c|c} \mathbf{0} & \sigma_a \\ \hline \sigma_a & \mathbf{0} \end{array}\right)\left(\begin{array}{c|c} \sigma_b & \mathbf{0} \\ \hline \mathbf{0} & -\sigma_b \end{array}\right) - \left(\begin{array}{c|c} \sigma_b & \mathbf{0} \\ \hline \mathbf{0} & -\sigma_b \end{array}\right)\left(\begin{array}{c|c} \mathbf{0} & \sigma_a \\ \hline \sigma_a & \mathbf{0} \end{array}\right) \\
&= \left(\begin{array}{c|c} \mathbf{0} & -(\sigma_a\sigma_b + \sigma_b\sigma_a) \\ \hline \sigma_a\sigma_b + \sigma_b\sigma_a & \mathbf{0} \end{array}\right) \\
&= \left(\begin{array}{c|c} \mathbf{0} & -2\mathbf{I}\delta_{ab} \\ \hline 2\mathbf{I}\delta_{ab} & \mathbf{0} \end{array}\right) = -2i\delta_{ab} D\,. \qquad (72)
\end{aligned}
$$

Alternatively, one can derive the commutation relations displayed in eq. (71) by employing the direct product representation of the Lie algebra generators given in eq. (70) and using $(A\otimes B)(C\otimes D) = AC\otimes BD$. For example, eq. (72) can also be obtained as follows.

$$
\begin{aligned}
[A_a, B_b] &= (\sigma_a\otimes\tau_1)(\sigma_b\otimes\tau_3) - (\sigma_b\otimes\tau_3)(\sigma_a\otimes\tau_1) \\
&= (\sigma_a\sigma_b)\otimes(\tau_1\tau_3) - (\sigma_b\sigma_a)\otimes(\tau_3\tau_1) \\
&= (\sigma_a\sigma_b)\otimes(-i\tau_2) - (\sigma_b\sigma_a)\otimes(i\tau_2) \\
&= (\sigma_a\sigma_b + \sigma_b\sigma_a)\otimes(-i\tau_2) \\
&= (2\mathbf{I}\delta_{ab})\otimes(-i\tau_2) = -2i\delta_{ab}\,\mathbf{I}\otimes\tau_2 = -2i\delta_{ab}D\,.
\end{aligned}
$$

All other commutation relations are easily derived using either of the methods shown above. Thus, the ten generators $\{A_a, B_a, C_a, D\}$ generate a Lie algebra, since the commutation relations close.

To determine the roots, we treat $\mathfrak{g}$ as a *complex* Lie algebra, so that we are free to consider complex linear combinations of generators. It is convenient to choose the Hermitian generators $H_1 = \sigma_3\otimes\mathbf{I} = C_3$ and $H_2 = \sigma_3\otimes\tau_3 = B_3$ to span the Cartan subalgebra. Indeed, these two generators are diagonal in the representation given in eq. (70). Therefore, the rank of the algebra $\mathfrak{g}$ is 2, corresponding to the maximal number of simultaneously diagonal generators.

We now rewrite the commutation relations given in eq. (71) in the Cartan-Weyl form. Starting from the commutation relations,

$$
[B_3, A_1] = [B_3, A_2] = 0\,, \qquad [C_3, A_1] = 2iA_2\,, \qquad [C_3, A_2] = -2iA_1\,,
$$

it is clear that we should define $A_\pm \equiv A_1 \pm iA_2$, in which case,

$$
[B_2, A_\pm] = 0\,, \qquad [C_3, A_\pm] = \pm 2A_\pm\,. \qquad (73)
$$

Next, we focus on the commutation relations,

$$
[B_3, A_3] = 2iD\,, \qquad [B_3, D] = -2iA_3\,, \qquad [C_3, A_3] = [C_3, D] = 0\,.
$$

These results motivate the definition $D_\pm \equiv A_3 \pm iD$, in which case,

$$
[B_3, D_\pm] = \pm 2D_\pm\,, \qquad [C_3, D_\pm] = 0\,. \qquad (74)
$$

The remaining commutation relations are:

$$[B_3, B_1] = 2iC_2 \,, \qquad [B_3, B_2] = -2iC_1 \,, \qquad [B_3, C_1] = 2iB_2 \,, \qquad [B_3, C_2] = -2iB_1 \,, \tag{75}$$

$$[C_3, B_1] = 2iB_2 \,, \qquad [C_3, B_2] = -2iB_1 \,, \qquad [C_3, C_1] = 2iC_2 \,, \qquad [C_3, C_2] = -2iC_1 \,. \tag{76}$$

Defining $B_\pm \equiv B_1 \pm iB_2$ and $C_\pm \equiv C_1 \pm iC_2$, eqs. (75) and (76) can be rewritten as:

$$[B_3, B_\pm] = \pm 2C_\pm \,, \qquad [B_3, C_\pm] = \pm 2B_\pm \,, \qquad [C_3, B_\pm] = \pm 2B_\pm \,, \qquad [C_3, C_\pm] = \pm 2C_\pm \,. \tag{77}$$

Thus, if we define $F_\pm \equiv B_\pm + C_\pm$ and $G_\pm \equiv B_\pm - C_\pm$, the eq. (77) will be in Cartan-Weyl form,

$$[B_3, F_\pm] = \pm 2F_\pm \,, \qquad [B_3, F_\pm] = \pm 2F_\pm \,, \qquad [C_3, G_\pm] = \mp 2G_\pm \,, \qquad [C_3, G_\pm] = \pm 2G_\pm \,. \tag{78}$$

To summarize, eqs. (73), (74) and (78) provide the Cartan-Weyl form for the commutation relations among the generators $H_i = \{C_3, B_3\}$ of the Cartan subalgebra and the off-diagonal generators $E_{\boldsymbol{\alpha}} \equiv \{A_\pm, D_\pm, E_\pm, F_\pm\}$. Note that we have chosen the generators to satisfy,

$$H_i^\dagger = H_i \,, \qquad E_{-\boldsymbol{\alpha}} = E_{\boldsymbol{\alpha}}^\dagger \,. \tag{79}$$

The root vectors are defined by the Cartan-Weyl form for the Lie algebra commutation relations, $[H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}}$, for $i = 1, 2, \ldots, r$, where $r = \text{rank}\ \mathfrak{g}$. In the present example, $r = 2$, $H_1 = C_3$, $H_2 = B_3$ and the off diagonal generators are $E_{\boldsymbol{\alpha}} \equiv \{A_\pm, D_\pm, E_\pm, F_\pm\}$. Hence, we identify the root vectors derived from the non-diagonal generators:

$$A_\pm : \quad \pm(0,2) \,, \qquad\qquad D_\pm : \quad \pm(2,0) \,, \tag{80}$$

$$F_\pm : \quad \pm(2,2) \,, \qquad\qquad G_\pm : \quad \pm(-2,2) \,, \tag{81}$$

where the first entry of the root vector is the eigenvalue of ad_{C_3} and the second entry of the root vector is the eigenvalue of ad_{B_3} The Cartan metric can be computed from the formula, $g_{ij} = \sum_{\boldsymbol{\alpha}} \alpha_i \alpha_j$. From the four root vectors obtained in eqs. (80) and (81), we immediately obtain

$$g_{ij} = 24\delta_{ij} \,. \tag{82}$$

The inverse Cartan metric is $g^{ij} = \frac{1}{24}\delta_{ij}$. One can now define the inner product on the root space,

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = g^{ij}\alpha_i\beta_j \,. \tag{83}$$

The squared-length of a root vector $\boldsymbol{\alpha}$ is given by

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = g^{ij}\alpha_i\alpha_j = \sum_{i=1}^{2} \alpha_i\alpha_i \,.$$

It is convenient to redefine the inner product given in eq. (83) by introducing an overall multiplicative positive constant such that the new inner product is Euclidean, $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_i \alpha_i\beta_i$.

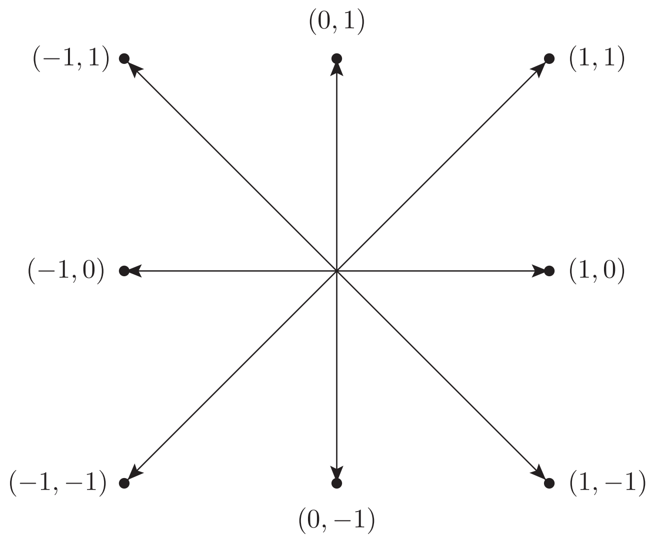

Figure 2: The root diagram for $\mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$.

Moreover, one is always free to rescale the generators of the Cartan subalgebra (which rescales the root vectors) in such a way that the shortest root vector has length 1. In these conventions, the rescaled roots are given by [cf. eqs. (80) and (81)], $\big\{\pm(0,1)\,,\,\pm(1,0)\,,\,\pm(1,1)\,,\,\pm(-1,1)\big\}$, and the corresponding root diagram is shown in Fig. 2 above, which we recognize as the root diagram for the rank-2 Lie algebra $\mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$.[8]

(b) Determine the simple roots and evaluate the corresponding Cartan matrix. Deduce the Dynkin diagram for this Lie algebra and identify it by name.

The simple roots correspond to the two smallest positive roots. These are

$$\boldsymbol{\alpha}_1 \equiv (0,1)\,, \quad \text{and} \quad \boldsymbol{\alpha}_2 \equiv (1,-1)\,. \tag{84}$$

It is a simple matter to check that the other two positive roots can be expressed as sums of simple roots,

$$(1,0) = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\,, \qquad\qquad (1,1) = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\,.$$

The Cartan matrix is defined by:[9]

$$A_{ij} = \frac{2(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)}\,, \tag{85}$$

[8]In the notation used here, $\mathfrak{sp}(n, \mathbb{C})$ is a Lie algebra of rank n. However, many books denote this Lie algebra by $\mathfrak{sp}(2n, \mathbb{C})$. Both conventions are common in the mathematics and physics literature.

[9]*Warning:* in the mathematics literature, eq. (85) is often employed as the definition of the transposed Cartan matrix. You should check carefully when using results from books on Lie algebras.

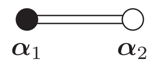

Figure 3: The Dynkin diagram for $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$.

where the inner product $(\boldsymbol{\alpha},\boldsymbol{\beta}) \equiv \sum_i \alpha_i\beta_i$ in the convention where $g_{ij} = \delta_{ij}$. Using eq. (84), we obtain $A_{11} = A_{22} = 2$, $A_{12} = -2$ and $A_{21} = -1$. That is,

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} . \tag{86}$$

The structure of the Dynkin diagram depends on the angle between the two simple roots:

$$\cos\varphi_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} = \frac{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)}{\sqrt{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_1)(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)}} = -\frac{1}{\sqrt{2}} .$$

Hence $\varphi_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} = 135^\circ$, which corresponds to a double line connecting the two balls of the Dynkin diagram. Hence, the Dynkin diagram corresponding to the Lie algebra, whose simple roots are given by eq. (84), is exhibited in Fig. 3, where the shaded ball corresponds to the simple root of the smallest length. In Cartan's notation, this Lie algebra is $B_2 \cong C_2$, which corresponds to $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$ as noted at the end of part (a).

(c) The fundamental weights $\boldsymbol{m}_i$ are defined in terms of the simple roots $\boldsymbol{\alpha}_j \in \Pi$ such that

$$\frac{2(\boldsymbol{m}_i,\boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_j)} = \delta_{ij} , \qquad \text{for } i,j = 1,2,\ldots,r , \tag{87}$$

where $r \equiv \text{rank}\,\mathfrak{g}$. Using the results of part (b), determine all the fundamental weights of $\mathfrak{g}$.

We can solve for the $\boldsymbol{m}_i$ by expanding the fundamental weight vectors in terms of the simple roots:

$$\boldsymbol{m}_i = \sum_{k=1}^{r} c_{ki}\boldsymbol{\alpha}_k .$$

Inserting this expression into eq. (87) yields,

$$\sum_{k=1}^{r} c_{ki}\frac{2(\boldsymbol{\alpha}_k,\boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_j)} = \delta_{ij} ,$$

which can be expressed in terms of the Cartan matrix A,

$$\sum_{k=1}^{r} c_{ki}A_{jk} = \delta_{ij} .$$

This implies that $c = A^{-1}$, and we conclude that

$$\boldsymbol{m}_i = \sum_{k=1}^{r}(A^{-1})_{ki}\boldsymbol{\alpha}_k . \tag{88}$$

Using the Cartan matrix given in eq. (86), the inverse is easily obtained:

$$A^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

Thus, eq. (88) yields the two fundamental weights of $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$,

$$\boldsymbol{m}_1 = \boldsymbol{\alpha}_1 + \tfrac{1}{2}\boldsymbol{\alpha}_2 = (\tfrac{1}{2}\,,\,\tfrac{1}{2})\,, \tag{89}$$

$$\boldsymbol{m}_2 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = (1\,,\,0)\,, \tag{90}$$

where we have used eq. (84) for the simple roots.

(d) Each of the r fundamental weights is the highest weight for an irreducible representation of $\mathfrak{g}$. Collectively, these are called the fundamental (or basic) representations of $\mathfrak{g}$. For each fundamental representation of $\mathfrak{g}$, compute the complete set of weights and draw the corresponding weight diagrams.[10] What are the corresponding dimensions of the fundamental representations of $\mathfrak{g}$.

The complete set of weights for the irreducible representations of $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$ corresponding to the highest weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively, can be obtained by the method of block weight diagrams described in Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Dover Publications, Inc., Mineola, NY, 2006).[11]

Given a highest weight $\boldsymbol{M}$, the corresponding *Dynkin labels* k_i (which are non-negative integers) are defined by

$$k_i \equiv \frac{2(\boldsymbol{M},\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_i)}\,, \qquad \text{where } \boldsymbol{\alpha}_i \in \Pi\,. \tag{91}$$

The irreducible representations of the Lie algebra $\mathfrak{g}$ are often denoted by placing the ith Dynkin label k_i above the ith ball of the Dynkin diagram (corresponding to the ith simple root $\boldsymbol{\alpha}_i$), as shown in Fig. 4 below.

The Dynkin labels for the fundamental weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are [cf. eq. (87)],[12]

$$\boldsymbol{m}_1:\ \ (1\,,\,0)\,, \qquad\qquad \boldsymbol{m}_2:\ \ (0\,,\,1)\,, \tag{92}$$

and the corresponding block weight diagrams are exhibited in Fig. 4.

The above block weight diagrams, corresponding to the two fundamental representations of $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$, were obtained as follows. We employed the theorem that establishes strings of weights of the form

$$\frac{2(\boldsymbol{m},\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_i)} - kA_{ji}, \qquad \text{for values of } k = 0, 1, 2, , \ldots, \frac{2(\boldsymbol{m},\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_i)}\,.$$

[10] The weight diagrams should be plotted on a two dimensional plane, where the axes correspond to the diagonalized generators normalized such that the shortest root vector has length 1.

[11] However, note that Cahn defines the Cartan matrix that is the transpose of our definition.

[12] Do not confuse the Dynkin labels of a weight with its coordinates in weight space given in eqs. (89) and (90). For example, the fundamental weight $\boldsymbol{m}_1 = \boldsymbol{\alpha}_1 + \frac{1}{2}\boldsymbol{\alpha}_2 = (\frac{1}{2}\,,\,\frac{1}{2})$, whereas its Dynkin labels are $(k_1, k_2) = (1\,,\,0)$, as indicated in eq. (92).

Figure 4: The block weight diagrams of the fundamental irreducible representations of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$.

Thus, starting with any weight $\boldsymbol{m}$, the Dynkin labels for the weights appearing below it in the block weight diagram are obtained by subtracting off the jth column of the Cartan matrix n times, where n is the jth positive Dynkin label of the weight.[13] Applying the above algorithm produces the Dynkin labels of the four weights corresponding to the representation specified by $\boldsymbol{m}_1$ and the five weights corresponding to representation specified by $\boldsymbol{m}_2$.

In this method, the computation of the multiplicity of a given weight requires additional analysis. But, for the simple cases treated above, all weights appear with multiplicity equal to one, in which case the dimension of the representation is simply equal to the number of weights in the block weight diagram.

Hence, the representations depicted by the block weight diagrams of Fig. 4 are four-dimensional and five-dimensional, respectively, The four-dimensional representation, corresponding to the highest weight $\boldsymbol{m}_1$, is precisely the matrix representation given in eq. (70). This is either the defining representation of $\mathfrak{sp}(2,\mathbb{C})$ or the spinor representation of $\mathfrak{so}(5,\mathbb{C})$.[14] In contrast, $\boldsymbol{m}_2$ is the highest weight of a five-dimensional representation, which corresponds to the defining representation of $\mathfrak{so}(5,\mathbb{C})$.

It is instructive to re-express the weights in terms of its coordinates in the vector space spanned by the simple roots. The weights can then be depicted as vectors in a two-dimensional plane. Given a weight specified by its Dynkin labels (k_1, k_2), the corresponding weight $\boldsymbol{m}$ is obtained by solving the equations [cf. eq. (91)]:

$$k_1 \equiv \frac{2(\boldsymbol{m}, \boldsymbol{\alpha}_1)}{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_1)}\,, \qquad\qquad k_2 \equiv \frac{2(\boldsymbol{m}, \boldsymbol{\alpha}_2)}{(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_2)}\,. \tag{93}$$

To solve for $\boldsymbol{m}$ in terms of k_1 and k_2, we expand $\boldsymbol{m}$ as a linear combination of simple roots [which are given explicitly in eq. (84)],

$$\boldsymbol{m} = c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2\,. \tag{94}$$

[13]If there are two (or more) positive Dynkin labels, then the block weight diagram branches. This does not occur in the examples exhibited in Fig. 4.

[14]Since $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$, the representations obtained above are representations of either Lie algebra. However, the interpretation of the representation depends on which choice of Lie algebra is made.

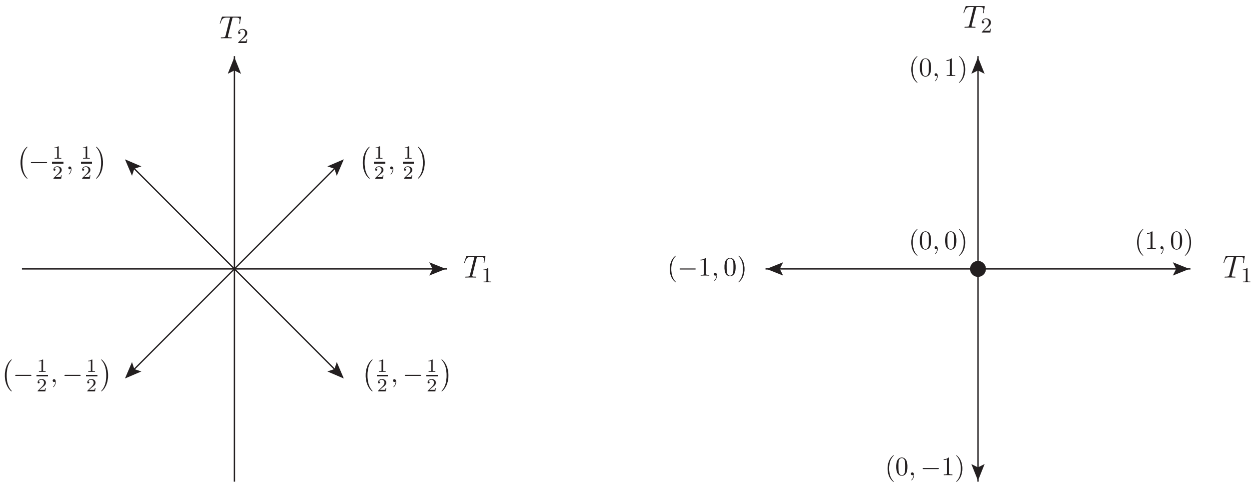

Figure 5: The weight diagrams of the fundamental representations of $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$, with dimensions four [left] and five [right], respectively.

Inserting this expression for $\boldsymbol{m}$ into eq. (93), it follows that:

$$
\begin{aligned}
k_1 &= \frac{2(c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2\,,\,\boldsymbol{\alpha}_1)}{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_1)} = 2c_1 - 2c_2\,,\\
k_2 &= \frac{2(c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2\,,\,\boldsymbol{\alpha}_2)}{(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)} = -c_1 + 2c_2\,,
\end{aligned}
$$

after using $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_1)=1$, $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)=-1$ and $(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)=2$. Solving for c_1 and c_2 then yields:

$$c_1 = k_1 + k_2\,, \qquad\qquad c_2 = \tfrac{1}{2}k_1 + k_2\,. \tag{95}$$

Hence, using eqs. (84) and (95), the weight $\boldsymbol{m}$ specified by eq. (94) is given by:

$$\boldsymbol{m} = \left(\tfrac{1}{2}k_1 + k_2\,,\,\tfrac{1}{2}k_1\right)\,. \tag{96}$$

As a check, if $\boldsymbol{m}=\boldsymbol{m}_1$ then $k_1=1$ and $k_2=0$, in which case $c_1=1$, $c_2=\frac{1}{2}$ and $\boldsymbol{m}_1=(\frac{1}{2}\,,\,\frac{1}{2})$ in agreement with eq. (89). Likewise, if $\boldsymbol{m}=\boldsymbol{m}_2$ then $k_1=0$ and $k_2=1$, in which case $c_1=c_2=1$ and $\boldsymbol{m}_1=(1\,,\,0)$ in agreement with eq. (90).

One can use eq. (96) to obtain the coordinates of all the weights exhibited in Fig. 4. For the four-dimensional representation specified by the Dynkin labels $(1,0)$ and the five-dimensional representation specified by the Dynkin labels $(0,1)$, the corresponding weight space diagrams are given in Fig. 5.[15] In particular, $T_1\equiv\frac{1}{2}H_1=\frac{1}{2}C_3$ and $T_2\equiv\frac{1}{2}H_2=\frac{1}{2}B_3$ are the diagonal generators normalized such that the shortest root vector has length 1. Given the explicit four-dimensional representation in eq. (70), one can check that the weight vectors, $\left\{(\frac{1}{2},\frac{1}{2})\,,\,(\frac{1}{2},-\frac{1}{2})\,,\,(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2})\right\}$, exhibited in Fig. 5 satisfy the eigenvalue equations,

$$T_i|\boldsymbol{m}\rangle = m_i|\boldsymbol{m}\rangle\,, \qquad \text{for } i=1,2\,, \tag{97}$$

where $\boldsymbol{m}=(m_1,m_2)$ are the coordinates in the T_1–T_2 plane.

[15] As previously noted, all weights shown in the two weight space diagrams above have multiplicity one, which means that the corresponding simultaneous eigenvector $|\boldsymbol{m}\rangle$ defined in eq. (97) is unique.

The weights of the five-dimensional representation, $\{(1,0),(0,1),(0,0),(0,-1),(-1,0)\}$, shown in Fig. 5 include a zero weight (indicated by the filled circle at the origin of the weight diagram). To check that eq. (97) is satisfied in this latter case, it is straightforward to construct explicit five-dimensional matrix representations of T_1 and T_2, which are the Cartan subalgebra generators in the defining representation of $\mathfrak{so}(5,\mathbb{C})$. Explicitly, we may choose the following Hermitian generators of the Cartan subalgebra,[16]

$$T_1 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{98}$$

The simultaneous normalized eigenvectors, denoted by $|\boldsymbol{m}\rangle$ in eq. (97), are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is now a simple matter to check that the weights of the five-dimensional representation shown in Fig. 5 satisfy eq. (97).

Finally, we note that the weight diagrams obtained above also apply to the real forms of $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$ [such as the corresponding compact real Lie algebras, $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$].

5. A basis for a three-dimensional complex Lie algebra $\mathfrak{g}$ satisfies the following commutation relations (where k is a nonzero real number):

$$[e_3, e_1] = e_1\,, \qquad [e_3, e_2] = ike_2\,, \qquad [e_1, e_2] = 0\,. \tag{99}$$

(a) Obtain the corresponding Cartan-Killing metric. Is $\mathfrak{g}$ semisimple? solvable? nilpotent?

We define the structure constants via

$$[e_i, e_j] = f_{ij}^{k} e_k\,. \tag{100}$$

Using eq. (99), we can identify the nonzero structure constants:

$$f_{13}^{1} = -f_{31}^{1} = -1\,, \qquad f_{23}^{2} = -f_{32}^{2} = -ik\,. \tag{101}$$

[16]The generators shown in eq. (98) are the obvious generalizations of the corresponding results of $\mathfrak{so}(3,\mathbb{C})$ and $\mathfrak{so}(4,\mathbb{C})$. In the case of $\mathfrak{so}(3,\mathbb{C})$, there is one Hermitian 3×3 matrix generator of the Cartan subalgebra in the defining representation (or equivalently the adjoint representation), usually denoted by $(T_3)_{jk} = -i\epsilon_{3jk}$. In the case of $\mathfrak{so}(4,\mathbb{C})$, there are two Hermitian 4×4 matrix generators of the Cartan subalgebra in the defining representation, denoted by $i\mathcal{A}_3$ and $i\mathcal{B}_3$ in the notation of problem 3 of this problem set.

The Cartan-Killing metric is defined in terms of the structure constants by:

$$g_{i\ell} = f_{ij}^{k} f_{\ell k}^{j}\,, \tag{102}$$

with an implicit sum over j and k. Hence, it follows that the nonzero values of $g_{i\ell}$ are

$$g_{33} = f_{31}^{1} f_{31}^{1} + f_{32}^{2} f_{32}^{2} = 1 - k^2\,. \tag{103}$$

Hence,

$$g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-k^2 \end{pmatrix}\,. \tag{104}$$

Since $\det g = 0$, it follows that $\mathfrak{g}$ is not semisimple.

To determine whether $\mathfrak{g}$ is solvable, we construct the derived series:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}\,,\,\mathfrak{g}], \quad \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \quad \text{etc.}$$

In light of eq. (99),

$$\mathfrak{g}^{(1)} = [\mathfrak{g}\,,\,\mathfrak{g}] = \begin{cases} \text{span}\{e_1, e_2\}, & k \neq 0, \\ \text{span}\{e_1\}, & k = 0. \end{cases} \tag{105}$$

and

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = 0\,. \tag{106}$$

Thus, the derived series terminate in two steps for any $k \in \mathbb{C}$. Hence, $\mathfrak{g}$ is a solvable Lie algebra.

To determine whether $\mathfrak{g}$ is solvable, we construct the lower central series:

$$\mathfrak{b}^{(0)} = \mathfrak{g}, \quad \mathfrak{b}^{(1)} = [\mathfrak{g}\,,\,\mathfrak{b}^{(0)}], \quad \mathfrak{b}^{(2)} = [\mathfrak{g}, \mathfrak{b}^{(1)}], \quad \text{etc.}$$

We again obtain eq. (105),

$$\mathfrak{b}^{(1)} = [\mathfrak{g}\,,\,\mathfrak{b}^{(0)}] = \begin{cases} \text{span}\{e_1, e_2\}, & k \neq 0, \\ \text{span}\{e_1\}, & k = 0. \end{cases} \tag{107}$$

However, in light of eq. (99),

$$\mathfrak{b}^{(2)} = [\mathfrak{g}\,,\,\mathfrak{b}^{(1)}] = \begin{cases} \text{span}\{e_1, e_2\}, & k \neq 0, \\ \text{span}\{e_1\}, & k = 0. \end{cases} \tag{108}$$

That is,

$$\mathfrak{b}^{(n)} = \mathfrak{b}^{(1)}\,, \quad \text{for all positive integers } n. \tag{109}$$

The lower central series never terminates at 0. Hence, we conclude that $\mathfrak{g}$ is *not* nilpotent for all $k \in \mathbb{C}$.

(b) Suppose that $k = 1$. Find the explicit form of the basis transformation that yields commutation relations of the generators such that all structure constants (with respect to the transformed basis) are real.

Some trial and error suggests defining

$$f_1 = e_1 + e_2\,, \qquad f_2 = i(e_2 - e_1)\,, \qquad f_3 = \alpha e_3\,, \tag{110}$$

where $\alpha \in \mathbb{C}$ to be determined. It then follows that

$$e_1 + ie_2 = \tfrac{1}{2}(1+i)(f_1 + f_2)\,, \qquad e_1 - ie_2 = \tfrac{1}{2}(1-i)(f_1 - f_2)\,. \tag{111}$$

A straightforward computation yields:

$$[f_1\,,\, f_2] = 0\,, \tag{112}$$

$$[f_1\,,\, f_3] = \alpha[e_1 + e_2\,,\, e_3] = -\alpha(e_1 + ie_2) = -\tfrac{1}{2}(1+i)\alpha(f_1 + f_2)\,, \tag{113}$$

$$[f_2\,,\, f_3] = i\alpha[e_2 - e_1\,,\, e_3] = i\alpha(e_1 - ie_2) = \tfrac{1}{2}(1+i)\alpha(f_1 - f_2)\,. \tag{114}$$

Thus, if we choose $\alpha = 1 - i$, then

$$[f_1\,,\, f_2] = 0\,, \tag{115}$$

$$[f_1\,,\, f_3] = -(f_1 + f_2)\,, \tag{116}$$

$$[f_2\,,\, f_3] = f_1 - f_2\,. \tag{117}$$

Thus, we have succeeded in finding a new basis where all the structure constants are real.

(c) Suppose instead that $k = 2$. Show that in this case, no change of basis exists in which all the structure constants (with respect to the transformed basis) are real.

Given the structure constants f_{ij}^k of a three-dimensional Lie algebra, it is convenient to define a 3×3 matrix A, whose matrix elements are given by[17]

$$A^{\ell k} = \tfrac{1}{2}\epsilon^{\ell ij} f_{ij}^k\,, \tag{118}$$

where $\epsilon^{ijk} = \epsilon_{ijk}$ is the usual Levi-Civita tensor. Given A, the structure constants are then given by

$$f_{ij}^k = \epsilon_{\ell ij} A^{\ell k}\,. \tag{119}$$

Although we will not need it here, one can show that since the structure constants satisfy the Jacobi identity, it follows that $\operatorname{adj} A$ is a symmetric matrix, where $\operatorname{adj} A$ is the adjugate of A (which is equal to the transpose of the cofactor matrix of A).

[17]See, e.g., E.M. Patterson, *Note on Three-Dimensional Lie Groups*, Glasgow Mathematical Journal, **2** no. 3, 112–115 (1955).

We showed in class that if we perform a basis change, $e'_i = e_j M^j{}_i$, where M is a nonsingular matrix, then the structure constants transform,

$$f'^{\,k}_{ij} = (M^{-1})^k{}_n M^\ell{}_i M^m{}_j f^n_{\ell m}\,. \tag{120}$$

Multiplying both sides of eq. (120) by $\frac{1}{2}\epsilon^{pij}$ and using eqs. (118) and (119),

$$A'^{pk} = \tfrac{1}{2}\epsilon^{pij}\epsilon_{q\ell m}(M^{-1})^k{}_n M^\ell{}_i M^m{}_j A^{qn}\,. \tag{121}$$

In light of the identity,

$$\epsilon_{q\ell m} M^\ell{}_i M^m{}_j = (\det M)\epsilon_{ijs}(M^{-1})^s{}_q\,, \tag{122}$$

and using $\epsilon^{pij}\epsilon_{ijs} = 2\delta^p_s$, we obtain

$$A'^{pk} = (\det M)(M^{-1})^p{}_q(M^{-1})^k{}_n A^{qn}\,. \tag{123}$$

Using the structure constants given by eq. (101), it follows that

$$A = \begin{pmatrix} 0 & -ik & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{124}$$

We would like to know if a nonsingular matrix M exists such that A' is a real matrix. Since the third row and column of A consists of zeros, it suffices to focus on the 2×2 matrix,

$$A_2 = \begin{pmatrix} 0 & -ik \\ 1 & 0 \end{pmatrix} \tag{125}$$

The 3×3 matrix M can be chosen to be

$$M = \begin{pmatrix} M_2 & 0 \\ 0 & c \end{pmatrix}, \tag{126}$$

where M_2 is a nonsingular 2×2 matrix and $c\det M_2 = \det M$. Denoting $S \equiv M_2^{-1}$, it follows that

$$A'_2 = (\det S)^{-1} S A_2 S^{\mathsf{T}}\,. \tag{127}$$

If $k = 0$, then the matrix A is trivially real, so we henceforth assume that $k \neq 0$. It is then convenient to define[18]

$$B \equiv A_2^{\mathsf{T}} A_2^{-1} = \begin{pmatrix} ik^{-1} & 0 \\ 0 & -ik \end{pmatrix}. \tag{128}$$

One can easily check that

$$B' = A_2'^{\mathsf{T}} A_2'^{-1} = SBS^{-1}\,. \tag{129}$$

If a basis exists in which A'_2 is real, then it follows that B' must also be real. The two eigenvalues of a real 2×2 matrix are either real or related by complex conjugation, since they are solutions to a quadratic equation with real coefficients. Moreover, the eigenvalues of B and B' are the same, since eigenvalues are preserved by a similarity transformation. Hence, it follows that B' can never be a real matrix if $ik^{-1} \neq ik^*$, or equivalently, $|k| \neq 1$.

[18] See, e.g., Roger A. Horn and Charles R. Johnson, *Matrix Analysis*, 2nd Edition (Cambridge University Press, Cambridge, UK, 2013) pp. 294–295. The matrix $B^{\mathsf{T}} = (A^{\mathsf{T}})^{-1}A$ is called the *cosquare* of the nonsingular matrix A.

In particular, if $k = 2$ no change of basis exists in which B' and hence A' is a real matrix. More generally, after using eq. (119), we conclude that if $|k| \neq 0, 1$, then no change of basis exists in which all the structure constants in eq. (100) are real.

<u>*REMARKS:*</u>

The analysis presented above can only be used to prove that no basis exists in which the matrix B' is real. In particular, if the eigenvalues of a 2×2 matrix B are complex conjugates of each other, then any matrix related to it via a similarity transformation may or may not be real. Indeed, complex matrices such as $B' = \mathrm{diag}(i\,, -i)$ exist whose eigenvalues are complex conjugates of each other. Thus, if $|k| = 1$ then the results obtained above imply only that the possibility exists in which B' and A' are real matrices.

Nevertheless, one can generalize the result of part (b) to show that if $|k| = 1$ then a change of basis does exist in which all the structure constants in eq. (100) are real. Denoting $k = e^{i\theta}$ (where $\theta \in \mathbb{R}$), we shall choose

$$f_1 = e_1 + e_2\,, \qquad f_2 = i(e_2 - e_1)\,, \qquad f_3 = \beta e_3\,, \tag{130}$$

where $\beta \equiv (1 - i)e^{-i\theta/2}$. A straightforward calculation yields:

$$[f_1\,,\, f_2] = 0\,, \tag{131}$$

$$[f_1\,,\, f_3] = \beta[e_1 + e_2\,,\, e_3] = -\beta(e_1 + ie^{i\theta}e_2) = -(1-i)\big(e^{-i\theta/2}e_1 + ie^{i\theta/2}e_2\big)\,, \tag{132}$$

$$[f_2\,,\, f_3] = i\beta[e_2 - e_1\,,\, e_3] = i\beta(e_1 - ie^{i\theta}e_2) = (1+i)\big(e^{-i\theta/2}e_1 - ie^{i\theta/2}e_2\big)\,. \tag{133}$$

Noting that $e_1 = \frac{1}{2}(f_1 + if_2)$ and $e_2 = \frac{1}{2}(f_1 - if_2)$, it follows that

$$[f_1\,,\, f_3] = -(f_1 + f_2)\cos(\theta/2) + (f_1 - f_2)\sin(\theta/2)\,, \tag{134}$$

$$[f_2\,,\, f_3] = (f_1 - f_2)\cos(\theta/2) + (f_1 + f_2)\sin(\theta/2)\,. \tag{135}$$

Thus, we have succeeded in finding a new basis where all the structure constants are real when $|k| = 1$. As expected, if $\theta = 0$, we recover the results of eqs. (116) and (117).

As a check of eqs. (134) and (135), we can use eqs. (123) and (124) with

$$M = \begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & (1-i)e^{-i\theta/2} \end{pmatrix}\,, \tag{136}$$

$$M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & (1+i)e^{i\theta/2} \end{pmatrix}\,, \tag{137}$$

$$\det M = 2(1+i)e^{-i\theta/2}\,, \tag{138}$$

to obtain

$$A' = \begin{pmatrix} \cos(\theta/2) + \sin(\theta/2) & -\cos(\theta/2) + \sin(\theta/2) & 0 \\ \cos(\theta/2) - \sin(\theta/2) & \cos(\theta/2) + \sin(\theta/2) & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{139}$$

Using eq. (119), we recover the results obtained in eqs. (134) and (135).