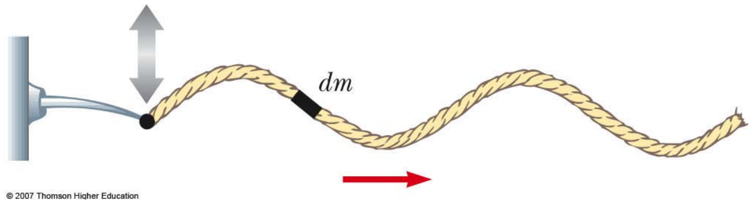
# Rate of Energy Transfer by Sinusoidal Waves on a String

Consider a sinusoidal wave traveling along the $x$-axis on a stretched string. Focus on the infinitesimal segment depicted by the blackened part of the rope (and labeled $dm$) in the figure below.




I shall compute the kinetic energy and the potential energy of this string segment due to the passage of a traveling wave. A nice reference to this topic, at a level slightly more advanced than Giancoli, is a book by A.P. French, *Vibrations and Waves* (W.W. Norton and Company, New York, 1971).

First, we compute the kinetic energy of the infinitesimal string segment. We set up our coordinate system in the usual way with the $x$-axis horizontal and the $y$-axis vertical. (In the notation of Giancoli, the vertical displacement $y$ is called $D$. In these notes, I shall use $y$ instead.) The element of string of mass $dm$ oscillates in the $y$-direction, undergoing simple harmonic motion. The kinetic energy of this string element is:

$$dK = \tfrac{1}{2}(dm)v_y^2\,,$$

where $dK$ is the infinitesimal kinetic energy of the infinitesimal string segment (of mass $dm$) and $v_y$ is the vertical velocity of the string segment transverse to the direction of propagation of the wave.

Before the wave passes through, the string is horizontal, and its mass is given by

$$dm = \mu dx\,,$$

where $\mu$ is the mass per unit length of the string. When the wave passes, the string bends and therefore stretches a little. But, we shall always work in the small angle approximation in which the angle the bent string makes with the $x$-axis is small. Thus, the correction to $dm = \mu dx$ is negligible and we shall ignore it.[1] Hence, combining the two equations above,

$$\frac{dK}{dx} = \tfrac{1}{2}\mu v_y^2\,.$$

The vertical displacement $y$ is assumed to have a sinusoidal form:[2]

$$y(x,t) = A\sin(kx-\omega t)\,,$$

where $\omega = 2\pi f$ and $k = 2\pi/\lambda$. Moreover,

$$v_y \equiv \frac{\partial y(x,t)}{\partial t} = -\omega A\cos(kx-\omega t)\,.$$

Thus, the kinetic energy per unit length (sometimes called the kinetic energy density) is

$$\boxed{\frac{dK}{dx} = \tfrac{1}{2}\mu\omega^2A^2\cos^2(kx-\omega t)\,.}$$

[1] The same approximation was made in deriving the one-dimensional wave equation.
[2] Again, remember that $y(x,t)$ is identical to what Giancoli calls $D(x,t)$.

It is convenient to write $dx = vdt$, where $v$ is the velocity of propagation of the wave (not to be confused with $v_y$ introduced above). Then, the above equation can be rewritten as:

$$\frac{dK}{dt} = \tfrac{1}{2}\mu v\omega^2 A^2 \cos^2(kx - \omega t)\,.$$

Typically, one is not interested in the instantaneous rate of change of kinetic energy. A more practical quantity is the kinetic energy per unit time integrated over a cycle of the wave, which shall be denoted by $\overline{K}$:

$$\overline{K} = \int_0^T \frac{dK}{dt}dt = \tfrac{1}{2}\mu v\omega^2 A^2 \int_0^T \cos^2(kx - \omega t)dt\,.$$

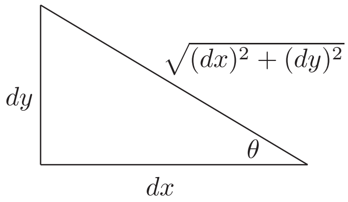
The integral over t is most easily performed by noting that one cycle of the function $\cos^2(kx - \omega t)$ looks exactly like one cycle of the function $\sin^2(kx - \omega t)$ excepted shifted by 90°. But clearly, the areas under these two functions integrated over a complete cycle are identical. Thus, we can write:

$$\begin{aligned}\int_0^T \cos^2(kx - \omega t)dt &= \frac{1}{2}\int_0^T \left[\cos^2(kx - \omega t) + \sin^2(kx - \omega t)\right] dt \\ &= \frac{1}{2}\int_0^T dt = \frac{T}{2}\,.\end{aligned}$$

Hence,

$$\boxed{\overline{K} = \tfrac{1}{4}\mu v\omega^2 A^2 T\,.}$$

We now examine the potential energy associated with the displacement of the string from equilibrium. Initially, the infinitesimal string is horizontal with a length $dx$. However, when the wave passes, the string is displaced vertically by an amount $dy$. As a result, the string is slightly stretched. In a diagram previously shown, the infinitesimal string element is at an angle, which can be approximated by a straight line whose length is $\sqrt{(dx)^2 + (dy)^2}$ as shown in the sketch below:




The string is stretched by an amount:

$$ds \equiv \sqrt{(dx)^2 + (dy)^2} - dx = dx\left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - 1\right].$$

From the diagram above $dy/dx = \tan\theta \simeq \theta$ in the small-angle approximation. This means that $(dy/dx)^2 \ll 1$, and therefore to a very good approximation,[3]

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \simeq 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2.$$

[3]To verify this approximation, square both sides of the equation and note that $(dy/dx)^4$ is extremely small and can be dropped. Better yet, try it on your calculator!

Consequently, the string is stretched by an amount

$$ds \simeq \tfrac{1}{2}dx \left(\frac{dy}{dx}\right)^2 .$$

The gain in potential energy $U$ is given by the work done to stretch the string by an amount $ds$ against a constant tension force $F_T$. That is,

$$dU = F_T ds = \tfrac{1}{2}F_T dx \left(\frac{dy}{dx}\right)^2 .$$

Using $y(x,t) = A\sin(kx - \omega t)$, we easily compute $dy/dx$, and conclude that

$$\frac{dU}{dx} = \tfrac{1}{2}F_T k^2 A^2 \cos^2(kx - \omega t) .$$

Recalling that the velocity of the transverse waves in a string is given by $v = \sqrt{F_T/\mu}$, we put $F_T = \mu v^2$ in the above equation. We also note that $v = f\lambda = \omega/k$, which allows us to write:

$$F_T k^2 = \mu v^2 k^2 = \mu\omega^2 .$$

Hence,

$$\boxed{\frac{dU}{dx} = \tfrac{1}{2}\mu\omega^2 A^2 \cos^2(kx - \omega t) .}$$

Remarkably, we see that

$$\boxed{\frac{dK}{dx} = \frac{dU}{dx}} .$$

That is, the instantaneous kinetic energy density is equal to the instantaneous potential energy density. Likewise, $dK/dt = dU/dt$ and $\overline{K} = \overline{U}$. Defining the total energy by

$$E \equiv K + U ,$$

we end up with:

$$\boxed{\overline{E} = \tfrac{1}{2}\mu v\omega^2 A^2 T .}$$

Finally, we define the power averaged over one cycle of the wave, $\overline{P}$, as follows:

$$\overline{P} \equiv \frac{1}{T}\int_0^T P(t)dt = \frac{1}{T}\int_0^T \left[\frac{dK}{dt} + \frac{dU}{dt}\right] = \frac{\overline{K} + \overline{U}}{T} = \frac{\overline{E}}{T} ,$$

where $P(t) \equiv dE/dt$ is the instantaneous power and $T$ is the period of the wave. Hence,

$$\overline{P} = \tfrac{1}{2}\mu\omega^2 A^2 v .$$

To make contact with eqs. (15-6) and (15-7) of Giancoli, we note that if the density of the string is $\rho$ and its cross-sectional area is $S$, then $\mu = \rho S$. Writing $\omega = 2\pi f$, we end up with

$$\boxed{\overline{P} = 2\pi^2 \rho S v f^2 A^2 ,}$$

and the intensity of the wave, defined by $I \equiv \overline{P}/S$, is given by

$$\boxed{I = 2\pi^2 v\rho f^2 A^2 .}$$

However, the derivation of these last two results given by Giancoli is hand-waving at best. It does not illustrate two important results obtained above: (i) the instantaneous kinetic and potential energy densities are equal, and (ii) the instantaneous kinetic and potential energy densities are sinusoidal in character (for a sinusoidal wave), and attain their maximum values when the displacement $y$ is *minimal* (and vice versa).[4]

[4] This is true, since when the displacement is minimal, both the transverse velocity of the string element $v_y$ and its slope $dy/dx$ are maximal. Likewise, when the displacement is maximal, the instantaneous transverse velocity is zero and the slope of the string is zero.