Evaluating integrals arising from Barr-Zee diagrams

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Abstract

In these notes, we provide an explicit calculation of some integrals that arises in the computation of the Barr-Zee Feynman diagrams [1] that contribute to the dipole moment operator.

Section 1: The function $g(z)$

Consider the integral [1, 2],

$$g(z) = \tfrac{1}{2}z \int_0^1 \frac{dx}{x(1-x)-z} \ln\left[\frac{x(1-x)}{z}\right], \qquad \text{for real } z > 0. \tag{1}$$

First we note that

$$x(1-x) - z = -(x-x_+)(x-x_-)\,, \tag{2}$$

where

$$x_\pm = \tfrac{1}{2}\bigl[1 \pm \sqrt{1-4z}\,\bigr]\,. \tag{3}$$

It follows that

$$x_+ + x_- = 1\,, \qquad\qquad x_+x_- = z\,. \tag{4}$$

Noting that for $z \neq \frac{1}{4}$,

$$\frac{1}{(x-x_+)(x-x_-)} = \frac{1}{x_+ - x_-}\left[\frac{1}{x-x_+} - \frac{1}{x-x_-}\right], \tag{5}$$

eq. (1) can be rewritten as:

$$g(z) = -\frac{z}{2\sqrt{1-4z}}\left\{\int_0^1 \frac{dx}{x-x_+}\ln\left[\frac{x(1-x)}{z}\right] - \int_0^1 \frac{dx}{x-x_-}\ln\left[\frac{x(1-x)}{z}\right]\right\}. \tag{6}$$

We first analyze the case of $0 < z < \frac{1}{4}$, in which case, $0 < x_- < x_+ < 1$. We shall employ the following result [3]:

$$\begin{aligned} G(y_1; b, c) &= \int_0^1 \frac{dy}{y-y_1}\ln\left(\frac{y^2+by+c-i\epsilon}{y_1^2+by_1+c}\right) \\ &= \mathrm{Li}_2\left(\frac{-y_1}{y_+-y_1}\right) + \mathrm{Li}_2\left(\frac{-y_1}{y_- - y_1}\right) - \mathrm{Li}_2\left(\frac{1-y_1}{y_+-y_1}\right) - \mathrm{Li}_2\left(\frac{1-y_1}{y_--y_1}\right). \end{aligned} \tag{7}$$

where y_1, b and c are real parameters, $y_\pm = \frac{1}{2}\bigl[-b \pm \sqrt{b^2-4c}\bigr]$, under the assumption that $0 \leq y_\pm \leq 1$ and $b^2 > 4c$, and the dilogarithm $\mathrm{Li}_2(z)$ is defined in the Appendix. Then,

$$g(z) = -\tfrac{1}{2}z\bigl[G(x_+, -1, 0) - G(x_-, -1, 0)\bigr]\,, \tag{8}$$

after noting that $x_\pm^2 - x_\pm = x_\pm(x_\pm - 1) = -x_\pm x_\mp = -z$.

We now make use of eq. (7) with $b = -1$, $c = 0$, and $y_1 = x_\pm$, which yields $y_+ = 1$ and $y_0 = 0$. It follows that

$$G(x_\pm, -1, 0) = \mathrm{Li}_2\left(\frac{-x_\pm}{1 - x_\pm}\right) - \mathrm{Li}_2\left(\frac{1 - x_\pm}{-x_\pm}\right) . \tag{9}$$

It is convenient to employ the identity [4]:

$$= \mathrm{Li}_2\left(\frac{-x}{1 - x}\right) = -\,\mathrm{Li}_2\,(x) - \tfrac{1}{2}\ln^2(1-x)\,, \qquad \text{for } x < 1. \tag{10}$$

Letting $x \to 1 - x$ yields a second identity:

$$\mathrm{Li}_2\left(\frac{1 - x}{-x}\right) = -\,\mathrm{Li}_2\,(1-x) - \tfrac{1}{2}\ln^2 x\,, \qquad \text{for } x > 0. \tag{11}$$

Hence,

$$G(x_\pm, -1, 0) = -\,\mathrm{Li}_2\,(x_\pm) + \mathrm{Li}_2\,(x_\mp) - \tfrac{1}{2}\ln^2 x_\mp + \tfrac{1}{2}\ln^2 x_\pm\,, \tag{12}$$

after using $x_\mp = 1 - x_\pm$. In particular,

$$G(x_-, -1, 0) = -G(x_+, -1, 0)\,. \tag{13}$$

We therefore end up with

$$\begin{aligned} g(z) = \frac{z}{\sqrt{1-4z}}\Big\{&\mathrm{Li}_2\big(\tfrac{1}{2}\big[1 + \sqrt{1-4z}\,\big]\big) - \mathrm{Li}_2\big(\tfrac{1}{2}\big[1 - \sqrt{1-4z}\,\big]\big) \\ &+ \tfrac{1}{2}\ln^2\big(\tfrac{1}{2}\big[1 - \sqrt{1-4z}\,\big]\big) - \tfrac{1}{2}\ln^2\big(\tfrac{1}{2}\big[1 + \sqrt{1-4z}\,\big]\big)\Big\}, \quad \text{for } 0 < z < \tfrac{1}{4}. \end{aligned} \tag{14}$$

One further simplification can be made by employing the identity [4]:

$$\mathrm{Li}_2(y) + \mathrm{Li}_2(1-y) = \tfrac{1}{6}\pi^2 - \ln y \ln(1-y)\,. \tag{15}$$

After making use of eqs. (4) and (15), we can rewrite eq. (14) in two equivalent forms:

$$\begin{aligned} g(z) &= \frac{z}{\sqrt{1-4z}}\Big\{2\,\mathrm{Li}_2\big(\tfrac{1}{2}\big[1 + \sqrt{1-4z}\,\big]\big) - \ln^2\big(\tfrac{1}{2}\big[1 + \sqrt{1-4z}\,\big]\big) + \tfrac{1}{2}\ln^2 z - \tfrac{1}{6}\pi^2\Big\} \\ &= \frac{z}{\sqrt{1-4z}}\Big\{-2\,\mathrm{Li}_2\big(\tfrac{1}{2}\big[1 - \sqrt{1-4z}\,\big]\big) + \ln^2\big(\tfrac{1}{2}\big[1 - \sqrt{1-4z}\,\big]\big) - \tfrac{1}{2}\ln^2 z + \tfrac{1}{6}\pi^2\Big\}, \\ &\qquad \text{for } 0 < z < \tfrac{1}{4}. \end{aligned} \tag{16}$$

It is instructive to evaluate the limit of $z \ll 1$. One can approximate $\sqrt{1-4z} \simeq 1 - 2z$ in the second form of $g(z)$ given in eq. (16) to obtain

$$g(z) \simeq z\big[\tfrac{1}{6}\pi^2 - 2\,\mathrm{Li}_2(z) + \tfrac{1}{2}\ln^2 z\big] = \tfrac{1}{2}z\big[\ln^2 z + \tfrac{1}{3}\pi^2\big] + \mathcal{O}(z^2)\,. \tag{17}$$

Next, we check the result for $z = \frac{1}{4}$. In this case, Mathematica yields:

$$g(z) = \ln 2 + \tfrac{2}{3}\big(4\ln 2 - 1\big)\big(z - \tfrac{1}{4}\big) + \mathcal{O}\big((z - \tfrac{1}{4})^2\big)\,. \tag{18}$$

One can verify the leading term of eq. (18) by evaluating eq. (1) for $z = \frac{1}{4}$. An explicit computation yields:

$$g\big(\tfrac{1}{4}\big) = -\frac{1}{2}\int_0^1 \frac{dx}{(2x-1)^2}\ln[4x(1-x)] = -\frac{1}{4}\int_{-1}^1 \frac{dy}{y^2}\ln(1-y^2) = \ln 2\,. \tag{19}$$

Next, we analyze the case of $z > \frac{1}{4}$. First, we can rewrite eq. (14) as

$$g(z) = \frac{z}{\sqrt{1-4z}}\left\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+\sqrt{1-4z}\,\big]\big) - \mathrm{Li}_2\big(\tfrac{1}{2}\big[1-\sqrt{1-4z}\,\big]\big) - \tfrac{1}{2}\ln z \ln\left(\frac{1+\sqrt{1-4z}}{1-\sqrt{1-4z}}\right)\right\}, \quad \text{for } 0 < z < \tfrac{1}{4}, \tag{20}$$

after noting that $\ln^2 x_- - \ln^2 x_+ = \ln(x_- x_+)\ln(x_-/x_+) = \ln z \ln(x_-/x_+)$. Moreover, eq. (20) remains valid for $z > \frac{1}{4}$ if we put $\sqrt{1-4z} = i\sqrt{4z-1}$. In light of $\mathrm{Li}_2(z^*) = [\mathrm{Li}_2(z)]^*$, it follows that:

$$\begin{aligned} g(z) &= \frac{2z}{\sqrt{4z-1}}\,\mathrm{Im}\Big\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+i\sqrt{4z-1}\,\big]\Big\} - \frac{z}{2i\sqrt{4z-1}}\ln z \ln\left(\frac{1+i\sqrt{4z-1}}{1-i\sqrt{4z-1}}\right) \\ &= \frac{2z}{\sqrt{4z-1}}\left[\mathrm{Im}\Big\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+i\sqrt{4z-1}\,\big]\Big\} - \tfrac{1}{2}\ln z\,\tan^{-1}\sqrt{4z-1}\,\right], \qquad \text{for } z > \tfrac{1}{4}. \end{aligned} \tag{21}$$

Using Ref. [4], one can express the imaginary part of the dilogarithm of a complex argument in terms of the Clausen function (which is defined in the Appendix):

$$\mathrm{Im}\,\mathrm{Li}_2(re^{i\theta}) = \omega\ln r + \tfrac{1}{2}\,\mathrm{Cl}_2(2\omega) - \tfrac{1}{2}\,\mathrm{Cl}_2(2\omega+2\theta) + \tfrac{1}{2}\,\mathrm{Cl}_2(2\theta)\,, \tag{22}$$

where

$$\omega \equiv \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right). \tag{23}$$

Note that if $re^{i\theta} = \frac{1}{2}\big[1+i\sqrt{4z-1}\,\big]$, then

$$r = \sqrt{z}\,, \qquad \sin\theta = \sqrt{1-\frac{1}{4z}}\,, \qquad \cos\theta = \frac{1}{2\sqrt{z}}\,. \tag{24}$$

Hence,

$$\theta = \omega = \tan^{-1}\sqrt{4z-1}\,, \tag{25}$$

where the principal value of the arctangent is used (i.e., $0 \le \tan^{-1}\sqrt{4z-1} \le \frac{1}{2}\pi$). It then follows that

$$\begin{aligned} &\mathrm{Im}\Big\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+i\sqrt{4z-1}\,\big]\Big\} - \tfrac{1}{2}\ln z\,\tan^{-1}\sqrt{4z-1} \\ &\qquad = \mathrm{Cl}_2\big(2\tan^{-1}\sqrt{4z-1}\big) - \tfrac{1}{2}\,\mathrm{Cl}_2\big(4\tan^{-1}\sqrt{4z-1}\big) \\ &\qquad = \mathrm{Cl}_2\big(\pi - 2\tan^{-1}\sqrt{4z-1}\,\big)\,, \end{aligned} \tag{26}$$

after employing the identity $\frac{1}{2}\,\mathrm{Cl}_2(4\theta) = \mathrm{Cl}_2(2\theta) - \mathrm{Cl}_2(\pi - 2\theta)$ given in Ref. [4]. Thus,

$$g(z) = \frac{2z}{\sqrt{4z-1}}\,\mathrm{Cl}_2\big(\pi - 2\tan^{-1}\sqrt{4z-1}\,\big)\,, \qquad \text{for } z > \tfrac{1}{4}. \tag{27}$$

Finally, in light of eqs. (24) and (25),

$$\theta = \tan^{-1}\sqrt{4z-1} = \cos^{-1}\frac{1}{2\sqrt{z}} = \frac{\pi}{2} - \sin^{-1}\frac{1}{2\sqrt{z}}\,, \tag{28}$$

where we have made use of the identity, $\sin^{-1} y + \cos^{-1} y = \frac{1}{2}\pi$, in the last step above. Hence, it follows that:

$$\pi - 2\tan^{-1}\sqrt{4z-1} = 2\sin^{-1}\frac{1}{2\sqrt{z}}\,. \tag{29}$$

Consequently, eq. (27) can be rewritten as:

$$g(z) = \frac{2z}{\sqrt{4z-1}}\,\mathrm{Cl}_2\biggl(2\sin^{-1}\frac{1}{2\sqrt{z}}\biggr)\,, \qquad\qquad \text{for } z > \tfrac{1}{4}. \tag{30}$$

It is instructive to compute $g\bigl(\frac{1}{4}\bigr)$. For this calculation, we shall employ an equivalent form [cf. eq. (26)]:

$$g(z) = \frac{2z}{\sqrt{4z-1}}\Bigl[\mathrm{Cl}_2\bigl(2\tan^{-1}\sqrt{4z-1}\bigr) - \tfrac{1}{2}\,\mathrm{Cl}_2\bigl(4\tan^{-1}\sqrt{4z-1}\bigr)\Bigr]\,. \tag{31}$$

Using the expansion (see eq. (4.28) of Ref. [4]),

$$\mathrm{Cl}_2(\theta) = \theta(1 - \ln|\theta|) + \tfrac{1}{72}\theta^3 + \mathcal{O}(\theta^5)\,, \tag{32}$$

and approximating $\tan^{-1}\sqrt{4z-1} \simeq \sqrt{4z-1}$, it follows that

$$g\bigl(\tfrac{1}{4}\bigr) = \ln\bigl(4\tan^{-1}\sqrt{4z-1}\bigr) - \ln\bigl(2\tan^{-1}\sqrt{4z-1}\bigr) = \ln 2\,, \tag{33}$$

in agreement with eq. (19).

As a second check, consider the limit where $z \gg 1$. Then,

$$2\sin^{-1}\frac{1}{2\sqrt{z}} = \frac{1}{\sqrt{z}}\left[1 + \frac{1}{24z} + \mathcal{O}\bigl(z^{-2}\bigr)\right]\,. \tag{34}$$

Using eq. (32), we end up with

$$g(z) = 1 + \tfrac{1}{2}\ln z + \frac{5 + 3\ln z}{36z} + \mathcal{O}(z^{-2})\,. \tag{35}$$

One can verify eq. (35) by taking the limit of eq. (1) for $z \gg 1$,

$$\begin{aligned} g(z) &= \tfrac{1}{2}\ln z - \int_0^1 \ln x\,dx - \frac{1}{z}\int_0^1 x(1-x)\log x + \frac{\ln z}{2z}\int_0^1 x(1-x)dx + \mathcal{O}(z^{-2}) \\ &= \left(\frac{1}{2} + \frac{1}{12z}\right)\ln z + 1 + \frac{5}{36z} + \mathcal{O}(z^{-2})\,. \end{aligned} \tag{36}$$

It is noteworthy that the value $g(1)$ can be expressed in terms of polygamma functions. Mathematica yields:

$$g(1) = \tfrac{1}{36}\Bigl[\psi_1\bigl(\tfrac{1}{6}\bigr) + \psi_1\bigl(\tfrac{1}{3}\bigr) - \psi_1\bigl(\tfrac{2}{3}\bigr) - \psi_1\bigl(\tfrac{5}{6}\bigr)\Bigr] \simeq 1.17195\,, \tag{37}$$

where

$$\psi_1(z) \equiv \frac{d^2}{dz^2}\ln\Gamma(z) = n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^2}\,. \tag{38}$$

Section 2: The function $f(z)$

Consider the integral [1,2],

$$f(z) = \tfrac{1}{2}z \int_0^1 \frac{1-2x(1-x)}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right]\,dx\,, \qquad \text{for real } z>0. \tag{39}$$

We can rewrite eq. (39) as

$$f(z) = (1-2z)g(z) - z\int_0^1 \ln\left[\frac{x(1-x)}{z}\right]\,dx = z(2+\ln z) + (1-2z)g(z)\,. \tag{40}$$

For $z \ll 1$, we see that $f(z) \simeq g(z) \simeq \frac{1}{2}z\ln^2 z$. Using eqs. (19) and (40), it follows that $f\left(\frac{1}{4}\right) = \frac{1}{2}$. For $z=1$, we have $f(1) = 2 - g(1) = 0.828046$. Finally, the leading contributions to $f(z)$ as $z\to\infty$ are obtained by employing eq. (35):

$$\begin{aligned} f(z) &= z(2+\ln z) + (1-2z)\left[1 + \tfrac{1}{2}\ln z + \frac{5+3\ln z}{36z} + \mathcal{O}(z^{-2})\right] \\ &= \tfrac{13}{18} + \tfrac{1}{3}\ln z + \mathcal{O}(z^{-1})\,. \end{aligned} \tag{41}$$

Section 3: The function $h(z)$

Consider the integral [2],

$$h(z) = -\tfrac{1}{2}z\int_0^1 \frac{dx}{x(1-x)-z}\left\{1 - \frac{z}{x(1-x)-z}\ln\left[\frac{x(1-x)}{z}\right]\right\}\,, \qquad \text{for real } z>0. \tag{42}$$

To evaluate $h(z)$, we first compute

$$\begin{aligned} \frac{d}{dz}\left(\frac{2g(z)}{z}\right) &= \frac{\partial}{\partial z}\int_0^1 \frac{dx}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right] \\ &= \int_0^1 \frac{dx}{\big[x(1-x)-z\big]^2}\,\ln\left[\frac{x(1-x)}{z}\right] - \frac{1}{z}\int_0^1 \frac{dx}{x(1-x)-z} \\ &= \frac{2h(z)}{z^2}\,. \end{aligned} \tag{43}$$

Hence, it follows that

$$h(z) = z^2\frac{d}{dz}\left(\frac{g(z)}{z}\right)\,. \tag{44}$$

We now use the results of Section 1. The following derivatives will be needed:

$$\frac{dx_\pm}{dz} = \mp\,(1-4z)^{-1/2}\,, \tag{45}$$

$$\frac{d}{dy}\,\mathrm{Li}_2(y) = -\frac{\ln(1-y)}{y}\,, \tag{46}$$

$$\frac{d}{d\theta}\,\mathrm{Cl}_2(\theta) = -\ln\big[2\sin\big(\tfrac{1}{2}\theta\big)\big]\,. \tag{47}$$

Using eq. (14), it follows that

$$\begin{aligned}\frac{d}{dz}\left(\frac{g(z)}{z}\right) &= \frac{2g(z)}{z(1-4z)}+\frac{1}{1-4z}\left\{\bigl[\ln x_+ + \ln x_-\bigr]\left(\frac{1}{x_+}+\frac{1}{x_-}\right)\right\} \\ &= \frac{2g(z)+\ln z}{z(1-4z)}\,, \qquad \text{for } 0<z<\tfrac{1}{4}. \end{aligned} \tag{48}$$

Likewise, using eq. (30), it follows that

$$\begin{aligned}\frac{d}{dz}\left(\frac{g(z)}{z}\right) &= -\frac{2g(z)}{z(4z-1)}-\frac{4}{\sqrt{4z-1}}\ln\Bigl[2\sin\Bigl(\sin^{-1}\frac{1}{2\sqrt{z}}\Bigr)\Bigr]\frac{d}{dz}\sin^{-1}\left(\frac{1}{2\sqrt{z}}\right) \\ &= -\frac{2g(z)}{z(4z-1)}+\frac{2\ln z}{\sqrt{4z-1}}\frac{1}{\sqrt{1-1/(4z)}}\bigl(-\tfrac{1}{4}z^{-3/2}\bigr)\,. \end{aligned} \tag{49}$$

Hence, we end up with:

$$\frac{d}{dz}\left(\frac{g(z)}{z}\right)=\frac{2g(z)+\ln z}{z(1-4z)}\,, \qquad \text{for } z>\tfrac{1}{4}. \tag{50}$$

Not surprisingly, the results obtained in eqs. (48) and (50) coincide (and thus are valid for all real values of $z>0$).

In light of eq. (44), it follows that

$$h(z)=\frac{z\bigl[2g(z)+\ln z\bigr]}{1-4z}\,. \tag{51}$$

Finally, we can check some limiting cases. Using eq. (17), it follows that for $z\ll 1$,

$$h(z)=z\ln z+z^2\bigl[\ln^2 z+4\ln z+\tfrac{1}{3}\pi^2\bigr]+\mathcal{O}(z^3)\,. \tag{52}$$

Using eqs. (37) and (51), $h(1)=-0.781302$. For $z\gg 1$, eq. (35) yields:

$$h(z)=-\tfrac{1}{2}\bigl(1+\ln z\bigr)-\frac{7+6\ln z}{36z}+\mathcal{O}(z^{-2})\,. \tag{53}$$

Using Mathematica, one can check eq. (53) by integrating the leading terms of $h(z)$ in the limit of $z\gg 1$. One can also verify either by explicit integration or by employing eqs. (18) and (51) that $h\bigl(\tfrac{1}{4}\bigr)=-\tfrac{1}{6}(1+2\ln 2)$. Note that the first order correction to this result requires a more accurate version of eq. (18):

$$g(z)=\ln 2+\tfrac{1}{6}(1-4z)(1-4\ln 2)-\tfrac{1}{60}(1-4z)^2(1+8\ln 2)+\mathcal{O}\bigl((1-4z)^3\bigr)\,. \tag{54}$$

Then,

$$\begin{aligned} 2zg(z) &= \tfrac{1}{2}\ln 2+\tfrac{1}{12}(1-10\ln 2)(1-4z)-\tfrac{1}{120}(11-32\ln 2)(1-4z)^2+\mathcal{O}\bigl((1-4z)^3\bigr)\,, \\ z\ln z &= -\tfrac{1}{2}\ln 2-\tfrac{1}{4}(1-2\ln 2)(1-4z)+\tfrac{1}{8}(1-4z)^2+\mathcal{O}\bigl((1-4z)^3\bigr)\,. \end{aligned} \tag{55}$$

Hence, eq. (51) yields

$$h(z)=-\tfrac{1}{6}(1+2\ln 2)+\tfrac{1}{30}(1+8\ln 2)(1-4z)+\mathcal{O}\bigl((1-4z)^2\bigr)\,. \tag{56}$$

Section 4: Summary of results

We define three integrals for real positive values of z [1, 2]:

$$g(z) = \tfrac{1}{2}z \int_0^1 \frac{dx}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right], \tag{57}$$

$$f(z) = \tfrac{1}{2}z \int_0^1 \frac{1-2x(1-x)}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right] dx\,, \tag{58}$$

$$h(z) = -\tfrac{1}{2}z \int_0^1 \frac{dx}{x(1-x)-z}\left\{1 - \frac{z}{x(1-x)-z}\ln\left[\frac{x(1-x)}{z}\right]\right\}. \tag{59}$$

Then, one can derive the following expressions for $f(z)$ and $h(z)$ in terms of $g(z)$:

$$f(z) = z(2+\ln z) + (1-2z)g(z)\,, \tag{60}$$

$$h(z) = \frac{z\big[2g(z)+\ln z\big]}{1-4z}\,. \tag{61}$$

An explicit expression for $g(z)$ is given by:

$$g(z) = \begin{cases} \dfrac{z}{\sqrt{1-4z}}\left\{\mathrm{Li}_2(x_+) - \mathrm{Li}_2(x_-) - \tfrac{1}{2}\ln z \ln\left(\dfrac{x_+}{x_-}\right)\right\}, & \text{for } 0 < z \leq \tfrac{1}{4}\,, \\[2ex] \dfrac{2z}{\sqrt{4z-1}}\,\mathrm{Cl}_2\left(2\sin^{-1}\dfrac{1}{2\sqrt{z}}\right), & \text{for } z > \tfrac{1}{4}\,, \end{cases} \tag{62}$$

where $x_\pm \equiv \frac{1}{2}\big[1 \pm \sqrt{1-4z}\,\big]$ and $0 \leq \sin^{-1}[1/(2\sqrt{z})] \leq \frac{1}{2}\pi$ (for $z \geq \frac{1}{4}$). In Fig. 1, we have employed Mathematica (Version 14.0) to produce plots of the functions $g(z)$, $f(z)$ and $-h(z)$ for $0.01 \leq z \leq 100$. This figure reproduces the results first shown in Fig. 3 of Ref. [2].

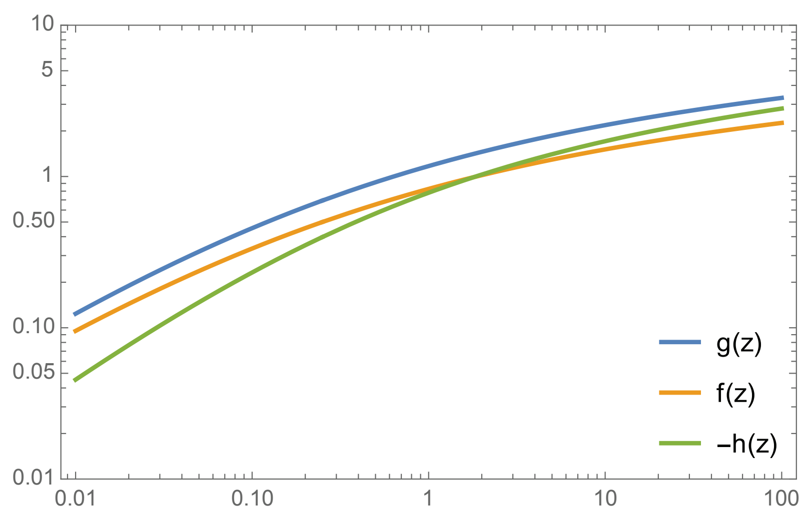

Figure 1: Plots of $g(z)$ given by eq. (62), $f(z)$ given by eq. (60), and $h(z)$ given by eq. (61) as a function of the variable z for $0.01 \leq z \leq 100$. These plots were produced using Version 14.0 of Mathematica.

Alternative expressions for $g(z)$ for $0 < z < \frac{1}{4}$ that involve only one dilogarithm can be found in eqs. (16) and (20). Note that the function $g(z)$ is continuous at $z = \frac{1}{4}$ with a value given by $g\big(\frac{1}{4}\big) = \ln 2$. Likewise, the function of $h(z)$ is also continuous (and finite) at $z = \frac{1}{4}$ with a value given by $h\big(\frac{1}{4}\big) = -\frac{1}{6}(1+2\ln 2)$. For $z \ll 1$, the leading behavior of the functions $f(z)$, $g(z)$, and $h(z)$ are given by: $f(z) \simeq g(z) \simeq \frac{1}{2}z\ln^2 z$ and $h(z) \simeq z\ln z$. For $z \gg 1$, the leading behavior of the functions are given by: $g(z) \simeq 1 + \frac{1}{2}\ln z$, $f(z) \simeq \frac{13}{18} + \frac{1}{3}\ln z$, and $h(z) \simeq -\frac{1}{2}(1 + \ln z)$. More accurate approximations for $g(z)$ and $h(z)$ are provided in Sections 1 and 3, respectively.

Appendix: Definitions of the dilogarithm and the Clausen function

In these notes, we follow the definitions given by Lewin in Ref. [4]. The dilogarithm is defined as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-w)}{w}\,dw\,, \tag{A.1}$$

where $z = x + iy$ is any complex number excluding $z = x \in \mathbb{R}$ for $1 < x < \infty$. By expanding $\ln(1 - w)$ in a power series around $w = 0$ and integrating term by term, one can derive a convergent series representation of the dilogarithm:

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}\,, \quad \text{for } |z| \leq 1. \tag{A.2}$$

It is conventional to define a single-valued dilogarithm function on the cut complex plane, where the branch cut is located on the real axis for $1 \leq x \leq \infty$. Although $z = x = 1$ is a branch point, the integral in eq. (A.1) is well defined there[1] and yields $\mathrm{Li}_2(1) = \frac{1}{6}\pi^2$. Lewin chooses to define $\ln(-x) = \lim_{\varepsilon\to 0}\ln(-x + i\varepsilon)$ for real $x > 0$ and $\mathrm{Li}_2(x) = \lim_{\varepsilon\to 0}\mathrm{Li}_2(x - i\varepsilon)$ for real $x > 1$, in which case $\mathrm{Im}\,\mathrm{Li}_2(x) = -\pi\ln(x)\Theta(x-1)$ for real $x > 1$, where the Heavyside step function is defined such that $\Theta(x - 1) = 1$ for $x > 1$ and $\Theta(x - 1) = 0$ for $x < 1$.[2]

The Clausen function is defined as

$$\mathrm{Cl}_2(\theta) = -\int_0^\theta \ln\big|2\sin\big(\tfrac{1}{2}\phi\big)\big|\,d\phi\,, \tag{A.3}$$

for $\theta \in \mathbb{R}$. A useful convergent series representation of the Clausen function is

$$\mathrm{Cl}_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}\,. \tag{A.4}$$

The Clausen function is an odd function of θ that satisfies the following periodic properties:

$$\mathrm{Cl}_2(2n\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\,\mathrm{Cl}_2(\theta)\,, \quad \text{for all integer values of } n, \tag{A.5}$$

and $\mathrm{Cl}_2(\pi + \theta) = -\,\mathrm{Cl}_2(\pi - \theta)$. Note that $\mathrm{Cl}_2(n\pi) = 0$ for any integer n, and $\mathrm{Cl}_2(\frac{1}{2}\pi) = G$, where G is Catalan's constant,

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \simeq 0.91596559\,. \tag{A.6}$$

[1] In light of eq. (A.2), $\mathrm{Li}_2(1) = \zeta(2) = \frac{1}{6}\pi^2$.

[2] It is now more common in the literature to define $\mathrm{Li}_2(x) = \lim_{\varepsilon\to 0}\mathrm{Li}_2(x + i\varepsilon)$ for real $x > 1$, although this choice of conventions affects none of the results presented in these notes.

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