Rational approximations of $\ln 2$

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Abstract

We examine a sequence of rational numbers $\{r_n\}$ that yield approximations of $\ln 2$ that become more precise as n increases. The approximation improves in accuracy by roughly an order of magnitude as n increases by one unit.

1. Introduction

There are many ways to obtain rational approximations of irrational numbers. In this short note, I will exploit a very clever method that is based on examining a sequence of integrals that is suitably chosen. The method is probably best known in association with obtaining rational approximations of π [1]. Indeed, the famous approximation that we all learned in high school, $\pi \approx 22/7$, is naturally obtained as the first of a sequence of ever improving approximations derived from examining the sequence of integrals introduced below. After a lightning review of the method for approximating π, I then apply the method, following Ref. [2], to obtain rational approximations of $\ln 2$.

Rational approximations of π can be obtained by considering the following sequence of integrals,

$$\mathcal{J}_n \equiv 4(-1)^n \int_0^1 \frac{dx}{1+x^2}\left(\frac{x^2(1-x)^2}{2}\right)^{2n} . \tag{1}$$

As shown in Ref. [1],

$$\mathcal{J}_n = \pi - p_n \,, \tag{2}$$

where p_n is a positive rational number for positive integer values of n. Since $0 \leq x(1-x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$, it follows that

$$0 < (-1)^n \mathcal{J}_n < \frac{1}{1024^n} \quad \Longrightarrow \quad \lim_{n\to\infty} p_n = \pi \,. \tag{3}$$

Hence for finite values of n, the sequence of rational numbers $\{p_n\}$ provides rational approximations of π with improving accuracy as n increases. Indeed, eq. (3) guarantees that the accuracy improves by more than three orders of magnitude as n is increased by one unit.

Remarkably, a closed form expression is given for p_n in Ref. [1],

$$p_n = \sum_{k=0}^{n-1} (-1)^k \frac{2^{4-2k}(4k)!(4k+3)!}{(8k+7)!}\big(820k^3 + 1533k^2 + 902k + 165\big) \,. \tag{4}$$

This result is derived by employing an identity given in Ref. [3] with no proof,

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = \big(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\big)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2}\,. \tag{5}$$

It is not very difficult to provide a derivation of eq. (5). We begin with the identity,

$$\frac{x^4(1-x)^4}{1+x^2} = P(x) + \frac{R}{1+x^2}\,, \tag{6}$$

for some polynomial $P(x)$ and constant R to be determined. To obtain R, we extend the function $x^4(1-x)^n/(1+x^2)$ into the complex plane. We then demand that the residue at the poles $\pm i$ are the same on both sides of eq. (6). This yields $R=-4$. It then follows that

$$P(x) = \frac{x^4(1-x)^4 + 4}{1+x^2}\,. \tag{7}$$

We can then use Mathematica to factor the numerator of $P(x)$,

$$x^4(1-x)^4 + 4 = (1+x^2)(4 - 4x^2 + 5x^4 - 4x^5 + x^6)\,. \tag{8}$$

Hence,

$$P(x) = x^6 - 4x^5 + 5x^4 - 4x^2 + 4\,. \tag{9}$$

Next, we multiply both sides of eq. (6) by $x^4(1-x)^4$. When carrying out the multiplication on the last term on the right hand side of eq. (6), we shall employ $x^4(1-x)^4 = (1+x^2)P(x) - 4$. The end result is

$$\frac{x^8(1-x)^8}{1+x^2} = \big[x^4(1-x)^4 - 4\big]P(x) + \frac{(-4)^2}{1+x^2}\,. \tag{10}$$

We again multiply both sides of eq. (10) by $x^4(1-x)^4$ and follow the same strategy as before to obtain,

$$\frac{x^{12}(1-x)^{12}}{1+x^2} = \big[x^8(1-x)^8 - 4x^4(1-x)^4 + (-4)^2\big]P(x) + \frac{(-4)^3}{1+x^2}\,. \tag{11}$$

Continuing the process, it should be clear that after n steps we arrive at

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = P(x)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2}\,. \tag{12}$$

Having identified $P(x)$ in eq. (9), we have indeed established the result quoted in eq. (5).

The evaluation of $\mathcal{J}_n$ is now straightforward, as the expression for p_n is expressed as the sum of integrals, each of which is recognized as the integral representation of a Beta function. In light of $\lim_{n\to\infty}\mathcal{J}_n = 0$, it follows that a rational approximation to π that monotonically improves in accuracy as n increases is given by $\pi \simeq p_n$, where p_n consists of the sum of the first n terms of the series given in eq. (4),

$$\pi = \frac{22}{7} - \frac{19}{15015} + \frac{543}{594914320} - \frac{77}{104187267600} + \cdots\,. \tag{13}$$

In the next section, we shall use a similar technique to obtain rational approximations of $\ln 2$.

2. Rational approximations of ln 2

Consider the following sequence of integrals [2],

$$\mathcal{I}_n \equiv (-1)^n \int_0^1 \frac{dx}{1+x} \left(\frac{x(1-x)}{2}\right)^n . \tag{14}$$

We will demonstrate below that

$$\mathcal{I}_n = \ln 2 - r_n \,, \tag{15}$$

where r_n is a positive rational number for positive integer values of n. Since $0 \leq x(1-x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$, it follows that

$$0 < (-1)^n \mathcal{I}_n < \frac{1}{8^n} \qquad \Longrightarrow \qquad \lim_{n\to\infty} r_n = \ln 2 \,. \tag{16}$$

Hence for finite values of n, the sequence of rational numbers $\{r_n\}$ provides rational approximations of $\ln 2$ with improving accuracy as n increases. Indeed, eq. (16) guarantees that the accuracy improves by roughly an order of magnitude as n is increased by one unit.

The relevant identity analogous to eq. (5) is derived using the same technique employed in Section 1. We first note that

$$\frac{x(1-x)}{1+x} = 2 - x - \frac{2}{1+x} \,. \tag{17}$$

Multiplying both sides of this equation by $x(1-x)$ and using $x(1-x) = (2-x)(1+x) - 2$ when multiplying the last term on the right hand side of eq. (17) yields,

$$\frac{x^2(1-x)^2}{1+x} = \bigl[x(1-x) - 2\bigr](2-x) + \frac{(-2)^2}{1+x} \,. \tag{18}$$

We again multiply by $x(1-x)$ and use $x(1-x) = (2-x)(1+x) - 2$ when multiplying the last term on the right hand side of eq. (17) to obtain

$$\frac{x^3(1-x)^3}{1+x} = \bigl[x^2(1-x)^2 - 2x(1-x) + (-2)^2\bigr](2-x) + \frac{(-2)^3}{1+x} \,. \tag{19}$$

Is should now be clear that after n steps, the end result is given by

$$\frac{x^n(1-x)^n}{1+x} = (2-x)\sum_{k=0}^{n-1} (-2)^{n-1-k} x^k (1-x)^k + \frac{(-2)^n}{1+x} \,. \tag{20}$$

Hence, it follows that

$$\mathcal{I}_n = \ln 2 + \sum_{k=0}^{n-1} (-2)^{-1-k} \int_0^1 x^k (1-x)^k (2-x)\, dx \,. \tag{21}$$

Using the integral expression of the Beta function,

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \frac{(r-1)!(s-1)!}{(r+s-1)!} = \int_0^1 x^{r-1}(1-x)^{s-1}\, dx \,, \tag{22}$$

it follows that

$$\mathcal{I}_n = \ln 2 + \sum_{k=0}^{n-1} (-2)^{-1-k} \left[2B(k+1,k+1) - B(k+2,k+1)\right] , \tag{23}$$

which can be simplified to obtain our final result,

$$\boxed{\mathcal{I}_n = \ln 2 - \frac{3}{4}\sum_{k=0}^{n-1} \frac{(-1)^k [k!]^2}{2^k (2k+1)!}} \tag{24}$$

Taking the limit as $n \to \infty$ and making use of $\lim_{n\to\infty} \mathcal{I}_n = 0$, we arrive at an interesting series expansion for $\ln 2$ (see the Appendix for an independent derivation),

$$\ln 2 = \frac{3}{4}\sum_{k=0}^{\infty} \frac{(-1)^k [k!]^2}{2^k (2k+1)!} = \frac{3}{4} - \frac{1}{16} + \frac{1}{160} - \frac{3}{4480} + \frac{1}{13440} - \frac{1}{118272} + \cdots . \tag{25}$$

It then follows that a good approximation to $\ln 2$ is given by the first n terms of eq. (25). That is, in eq. (15) r_n is given by,

$$r_n = \frac{3}{4}\sum_{k=0}^{n-1} \frac{(-1)^k [k!]^2}{2^k (2k+1)!} . \tag{26}$$

In Table 1 we list the individual terms, a_k, that appear in eq. (25) and the corresponding values of $r_n \equiv \sum_{k=0}^{n-1} a_k$ for positive integer values of $n \leq 11$. Ten digit accuracy is obtained for $n = 11$. Note that the accuracy of the approximation to $\ln 2$ improves by roughly an order of magnitude with each increase of n by one unit, as anticipated.

n	a_{n-1}	r_n	numerical value
1	$\frac{3}{4}$	$\frac{3}{4}$	0.75
2	$-\frac{1}{16}$	$\frac{11}{16}$	0.6875
3	$\frac{1}{160}$	$\frac{111}{160}$	0.69375
4	$-\frac{3}{4480}$	$\frac{621}{896}$	0.69308035714
5	$\frac{1}{13440}$	$\frac{2329}{3360}$	0.69315476190
6	$-\frac{1}{118272}$	$\frac{19519}{28160}$	0.69314630682
7	$\frac{1}{1025024}$	$\frac{3552463}{5125120}$	0.69314728241
8	$-\frac{1}{8785920}$	$\frac{42629549}{61501440}$	0.69314716859
9	$\frac{1}{74680320}$	$\frac{241567449}{348508160}$	0.69314718198
10	$-\frac{3}{1891901440}$	$\frac{834505731}{1203937280}$	0.69314718039
11	$\frac{1}{5297324032}$	$\frac{18359126087}{26486620160}$	0.69314718058

Table 1: Rational approximations, $r_n = \sum_{k=0}^{n-1} a_k$, of $\ln 2$ obtained using Mathematica. The approximations become more accurate as n increases. With eleven digit accuracy, $\ln 2 \simeq 0.69314718056$.

APPENDIX

In this Appendix, an independent derivation of eq. (25) is provided [4]. We first employ eq. (16) on p. 102 of Ref. [5],

$$ {}_2F_1(\tfrac{1}{2}, 1, \tfrac{3}{2}; z^2) = \frac{1}{2z} \ln\left(\frac{1+z}{1-z}\right) , \tag{27} $$

where ${}_2F_1$ is the Gauss hypergeometric function. We next use eq. (4) on p. 105 of Ref. [5],

$$ {}_2F_1(a, b; c; z) = (1-z)^{-b}\, {}_2F_1\bigl(c-a, b; c, z/(z-1)\bigr) , \tag{28} $$

to obtain,

$$ \ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1-z^2}\, {}_2F_1\bigl(1, 1; \tfrac{3}{2}, z^2/(z^2-1)\bigr) . \tag{29} $$

Setting $z = \frac{1}{3}$ yields,

$$ \ln 2 = \tfrac{3}{4}\, {}_2F_1\bigl(1, 1; \tfrac{3}{2}, -\tfrac{1}{8}\bigr) . \tag{30} $$

Using the series representation of the Gauss hypergeometric function given by eqs. (1) and (2) on p. 101 of Ref. [5],

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!}\, z^k , \tag{31} $$

it follows that

$$ \ln 2 = \tfrac{3}{4}\Gamma\bigl(\tfrac{3}{2}\bigr) \sum_{k=0}^{\infty} \frac{k!}{2^{3k}\Gamma\bigl(\tfrac{3}{2}+k\bigr)} . \tag{32} $$

Finally, using $\Gamma\bigl(\frac{3}{2}\bigr) = \frac{1}{2}\sqrt{\pi}$ and making use of the duplication formula (see, e.g., eq. (1.2.3) of Ref. [6]),

$$ \Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n}\,n!} , \tag{33} $$

we arrive at our final result,

$$ \ln 2 = \frac{3}{4} \sum_{k=0}^{\infty} \frac{(-1)^k [k!]^2}{2^k (2k+1)!} , \tag{34} $$

in agreement with eq. (25).

References

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