The differential operator $\vec{\boldsymbol{L}}$ and the Debye Decomposition Theorem

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March 25, 2026

Abstract

The differential operator $\vec{\boldsymbol{L}} \equiv -i\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}}$ appears both in the multipole expansion of the radiation fields in classical electrodynamics (via the Debye decomposition theorem) and in the quantum theory of angular momentum. In these notes, the Debye decomposition theorem is stated and proved, and properties of the differential operator $\vec{\boldsymbol{L}}$ are elucidated.

The differential operator $\vec{\boldsymbol{L}}$ is defined as[1]

$$\vec{\boldsymbol{L}} \equiv -i\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}} = i\left(\hat{\boldsymbol{\theta}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\right)\,, \tag{1}$$

where we have written out the explicit form in spherical coordinates (where θ is the polar angle and ϕ is the azimuthal angle) with respect to the spherical basis. In particular, if $\vec{\boldsymbol{x}} = r\hat{\boldsymbol{n}}$, where $r = |\vec{\boldsymbol{x}}|$, then $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta$. In these notes, we will employ the more common notation where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$.

It is sometimes useful to convert between the Cartesian basis and the spherical basis. For example,

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi\,, \tag{2}$$

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi\,, \tag{3}$$

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\,. \tag{4}$$

Inverting these results yields

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta\,, \tag{5}$$

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}}\cos\theta\cos\phi + \hat{\boldsymbol{y}}\cos\theta\sin\phi - \hat{\boldsymbol{z}}\sin\theta\,, \tag{6}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi\,. \tag{7}$$

The differential operator $\vec{\boldsymbol{L}}$ plays a critical role in Debye's decomposition theorem, which is stated below.[2]

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[1] Note that in quantum mechanics, the angular momentum operator in the coordinate representation is given by $\hbar\vec{\boldsymbol{L}}$.

[2] See, e.g., Dietman Petrascheck and Franz Schwabl, *Electrodynamics* (Springer Nature, Berlin, Germany, 2025) p. 151. A proof of the first part of this theorem is given in C.G. Gray and B. Nickel, *Debye potential representation of vector fields*, American Journal of Physics **46**, 735–736 (1978). The properties of the operator $\vec{\boldsymbol{L}}$ are discussed in Appendix F of C.G. Gray, *Multipole expansions of electromagnetic fields using Debye potentials*, American Journal of Physics **46**, 169–179 (1978).

Theorem (**Debye decomposition**). *Let $\vec{\boldsymbol{F}}$ be a divergenceless vector field, $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{F}}=0$ (also called a solenoidal field). Then, there exist scalar functions ψ and χ (called the Debye potentials) such that*

$$\vec{\boldsymbol{F}}=\vec{\boldsymbol{L}}\psi+(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}})\chi\,, \tag{8}$$

where the Debye potentials are unique up to an arbitrary radial function. That is, $\vec{\boldsymbol{F}}$ is unchanged under the transformations,

$$\psi(\vec{\boldsymbol{x}})\to\psi(\vec{\boldsymbol{x}})+f(r)\,,\qquad\qquad \chi(\vec{\boldsymbol{x}})\to\chi(\vec{\boldsymbol{x}})+g(r)\,, \tag{9}$$

for arbitrary radial functions $g(r)$ and $g(r)$.

If in addition, $\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})$ satisfies the homogeneous Helmholtz equation, $(\vec{\boldsymbol{\nabla}}^{\,2}+k^2)\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})=0$, then one can adjust the radial functions $f(r)$ and $g(r)$ such that

$$(\vec{\boldsymbol{\nabla}}^{\,2}+k^2)\psi(\vec{\boldsymbol{x}})=(\vec{\boldsymbol{\nabla}}^{\,2}+k^2)\chi(\vec{\boldsymbol{x}})=0\,. \tag{10}$$

For completeness, it is useful to note that if the condition $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{F}}=0$ is removed, then an arbitrary vector field $\vec{\boldsymbol{F}}$ can be represented in terms of three scalar fields as

$$\vec{\boldsymbol{F}}=\vec{\boldsymbol{L}}\psi+(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}})\chi+\vec{\boldsymbol{\nabla}}\phi\,, \tag{11}$$

where the scalar field ϕ is determined by solving the Poisson equation, $\vec{\boldsymbol{\nabla}}^2\phi=\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{F}}$, which is obtained by noting that $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{L}}=\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}})=0$ [cf. eqs. (20) and (21) below].

Before providing a proof of the Debye decomposition theorem above, it will be useful to collect many useful results and identities related to the differential operator $\vec{\boldsymbol{L}}$. First, we record the following representations:

$$\vec{\boldsymbol{L}}=i\left(\hat{\boldsymbol{\theta}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi}-\hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\right)\,, \tag{12}$$

$$L_z=-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-i\frac{\partial}{\partial\phi}\,, \tag{13}$$

$$L_\pm\equiv L_x\pm iL_y=e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta}+i\cot\theta\,\frac{\partial}{\partial\phi}\right)\,, \tag{14}$$

$$\vec{\boldsymbol{L}}^{\,2}=L_x^2+L_y^2+L_z^2=-\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\,, \tag{15}$$

$$-i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{L}}=r\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta}+\frac{\hat{\boldsymbol{\phi}}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\,, \tag{16}$$

$$-i\,\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}=\frac{\hat{\boldsymbol{r}}}{r}\,\vec{\boldsymbol{L}}^{\,2}+\hat{\boldsymbol{\theta}}\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial r\partial\theta}\right)+\frac{\hat{\boldsymbol{\phi}}}{\sin\theta}\left(\frac{1}{r}\,\frac{\partial}{\partial\phi}+\frac{\partial^2}{\partial r\partial\phi}\right)\,. \tag{17}$$

The operators $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{L}}^{\,2}$ are purely angular operators. Moreover, as a consequence of eqs. (12), (15) and (17), it follows that for any radial function $f(r)$,

$$\vec{\boldsymbol{L}}f(r)=\vec{\boldsymbol{L}}^{\,2}f(r)=\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\,f(r)=0\,. \tag{18}$$

Next, we record the following useful operator identities:

$$\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{L}} = 0, \tag{19}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{L}} = \vec{\boldsymbol{L}} \cdot \vec{\boldsymbol{\nabla}} = 0, \tag{20}$$

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{L}}) = 0, \tag{21}$$

$$\vec{\boldsymbol{L}} \cdot (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{L}}) = 0, \tag{22}$$

$$\vec{\boldsymbol{L}}\,\vec{\boldsymbol{L}}^{\,2} = \vec{\boldsymbol{L}}^{\,2}\vec{\boldsymbol{L}}, \tag{23}$$

$$\vec{\boldsymbol{L}}\,\vec{\boldsymbol{\nabla}}^{\,2} = \vec{\boldsymbol{\nabla}}^{\,2}\vec{\boldsymbol{L}}, \tag{24}$$

$$\vec{\boldsymbol{\nabla}} = \frac{\vec{\boldsymbol{x}}}{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{L}}, \tag{25}$$

$$\vec{\boldsymbol{\nabla}}^{\,2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{\boldsymbol{L}}^{\,2}}{r^2}, \tag{26}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{L}} = -i\vec{\boldsymbol{x}}\,\vec{\boldsymbol{\nabla}}^{\,2} + i\vec{\boldsymbol{\nabla}}\left(1 + r\frac{\partial}{\partial r}\right), \tag{27}$$

$$\vec{\boldsymbol{x}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{L}}) = i\vec{\boldsymbol{L}}^{\,2}, \tag{28}$$

$$\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{L}}) = -\vec{\boldsymbol{L}}\left(1 + r\frac{\partial}{\partial r}\right), \tag{29}$$

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{L}}) = -\vec{\boldsymbol{\nabla}}^{\,2}\vec{\boldsymbol{L}}, \tag{30}$$

$$\vec{\boldsymbol{L}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{L}}) = 0. \tag{31}$$

In particular, the differential operators L_i and L_j do not commute (for $i \neq j$). Instead, they satisfy the following commutation relations,

$$\big[L_i\,,\,L_j\big] \equiv L_iL_j - L_jL_i = i\epsilon_{ijk}L_k\,, \tag{32}$$

with an implicit sum over the repeated index k, where $L_1 \equiv L_x$, $L_2 \equiv L_y$, and $L_3 \equiv L_z$. Eq. (32) is equivalent to the equation $\epsilon_{ijk}L_iL_j = iL_k$, which can be rewritten in vector notation as

$$\vec{\boldsymbol{L}} \times \vec{\boldsymbol{L}} = i\vec{\boldsymbol{L}}\,. \tag{33}$$

Likewise, the differential operators L_i do not commute with the coordinates x_j (for $i \neq j$):

$$\big[L_i\,,\,x_j\big] \equiv L_ix_j - x_jL_i = i\epsilon_{ijk}x_k\,, \qquad \text{or equivalently, } \vec{\boldsymbol{L}} \times \vec{\boldsymbol{x}} = i\vec{\boldsymbol{x}}. \tag{34}$$

Eqs. (19)–(34) should be understood as operator equations that act on a function $f(\vec{\boldsymbol{x}})$. The following additional identities are also noteworthy:

$$\vec{\boldsymbol{L}} \cdot \vec{\boldsymbol{F}} = -i\vec{\boldsymbol{x}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}), \tag{35}$$

$$\vec{\boldsymbol{L}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) = i\left[\vec{\boldsymbol{\nabla}}^{\,2}\big(\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{F}}\big) + \left(2 + r\frac{\partial}{\partial r}\right)\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}\right]. \tag{36}$$

Finally, we note that the spherical harmonics, $Y_{\ell m}(\theta,\phi)$ are simultaneous eigenfunctions of $\vec{\boldsymbol{L}}^{\,2}$ and L_z,

$$\vec{\boldsymbol{L}}^{\,2} Y_{\ell m}(\theta,\phi) = \ell(\ell+1) Y_{\ell m}(\theta,\phi)\,, \tag{37}$$

$$L_z Y_{\ell m}(\theta,\phi) = m Y_{\ell m}(\theta,\phi)\,, \tag{38}$$

where $\ell = 0, 1, 2, 3, \ldots$, and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. In addition, the operators $L_\pm$ [cf. eq. (14)], when acting on the spherical harmonics, yield

$$L_+ Y_{\ell m}(\theta,\phi) = \sqrt{(\ell-m)(\ell+m+1)}\, Y_{\ell,m+1}(\theta,\phi)\,, \tag{39}$$

$$L_- Y_{\ell m}(\theta,\phi) = \sqrt{(\ell+m)(\ell-m+1)}\, Y_{\ell,m-1}(\theta,\phi)\,, \tag{40}$$

which is why the $L_\pm$ are called raising and lowering operators, respectively.

Returning to the Debye's decomposition theorem, we note that the solenoidal field $\vec{\boldsymbol{F}}$ is unchanged under the transformations specified by eq. (9) as a consequence of eq. (18). Moreover, with $\vec{\boldsymbol{F}}$ given by eq. (8), we may use eqs. (19), (28) and (31) to obtain:

$$\vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = i\vec{\boldsymbol{L}}^{\,2} \chi(\vec{\boldsymbol{x}})\,, \tag{41}$$

$$\vec{\boldsymbol{L}} \boldsymbol{\cdot} \vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{L}}^{\,2} \psi(\vec{\boldsymbol{x}})\,, \tag{42}$$

Using eq. (35), it follows that eq. (42) is equivalent to

$$\vec{\boldsymbol{x}} \boldsymbol{\cdot} (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) = i\vec{\boldsymbol{L}}^{\,2} \psi(\vec{\boldsymbol{x}})\,. \tag{43}$$

Expanding $\psi(\vec{\boldsymbol{x}})$ and $\chi(\vec{\boldsymbol{x}})$ in spherical harmonics, one can then solve for the Debye potentials. Note that the expansion in spherical harmonics starts at $\ell = 1$, since the $\ell = 0$ term is constant and thus is annihilated by the operator $\vec{\boldsymbol{L}}^{\,2}$. That is $\vec{\boldsymbol{F}}$ is determined by the radial components of $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}$, respectively [cf. eqs. (41) and (43)].

Thus, to finish the proof of the first part of Debye's decomposition theorem, one must show that it is complete. That is, no further terms need appear on the right hand side of eq. (8). Here, we quote the proof provided by Gray and Nickel cited in the second reference of footnote 2. To show that no further terms are required on the right hand side of eq. (8), we show that if $\vec{\boldsymbol{F}}$ has the three properties: (i) $\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{F}} = 0$, (ii) $\hat{\boldsymbol{r}} \boldsymbol{\cdot} \vec{\boldsymbol{F}} = 0$, and (iii) $\hat{\boldsymbol{r}} \boldsymbol{\cdot} (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) = 0$, then $\vec{\boldsymbol{F}} = 0$. Consider the field $\vec{\boldsymbol{F}}$ to be defined in a region of space between two concentric spheres of radii a and b. On a sphere S of radius r (where $a < r < b$), the lines of $\vec{\boldsymbol{F}}$ are tangential [property (ii)], do not cross ($\vec{\boldsymbol{F}}$ is unique at every point), and do not end [property (i)]. Thus the lines of $\vec{\boldsymbol{F}}$ form closed loops on S. But closed loops on S imply a component of the curl in the radial direction, contradicting property (iii). Hence $\vec{\boldsymbol{F}} = 0$.

We now examine the second part of Debye's decomposition theorem by supposing that the solenoidal field $\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})$ also satisfies the Helmholtz equation,

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = 0\,. \tag{44}$$

Then, it follows that

$$\vec{\boldsymbol{\nabla}}^{\,2}\big(\vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{F}}\big) = \vec{\boldsymbol{x}} \boldsymbol{\cdot} \big(\vec{\boldsymbol{\nabla}}^{\,2} \vec{\boldsymbol{F}}\big) + 2\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{F}} = -k^2 \vec{\boldsymbol{x}} \boldsymbol{\cdot} \vec{\boldsymbol{F}}\,, \tag{45}$$

after using $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} = 0$ and making use of eq. (44). Hence, we obtain

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{F}} = 0\,. \tag{46}$$

It then follows from eqs. (24) and (41) that

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\vec{\boldsymbol{L}}^{\,2}\chi(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{L}}^{\,2}(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\chi(\vec{\boldsymbol{x}}) = 0\,. \tag{47}$$

In light of eq. (18), we can conclude that

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\chi(\vec{\boldsymbol{x}}) = h(r)\,, \tag{48}$$

for some radial function $h(r)$. If we now transform $\chi(\vec{\boldsymbol{x}}) \to \chi'(\vec{\boldsymbol{x}}) = \chi(\vec{\boldsymbol{x}}) + g(r)$ as specified in eq. (9) and choose $g(r)$ such that

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)g(r) = -h(r)\,, \tag{49}$$

then it follows that

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\chi'(\vec{\boldsymbol{x}}) = (\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\big[\chi(\vec{\boldsymbol{x}}) + g(r)\big] = h(r) - h(r) = 0\,. \tag{50}$$

In fact, one can always find a function $g(r)$ such that eq. (49) is satisfied. In particular, the Green function that is used in the solution to the inhomogeneous Helmholtz equation satisfies:

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\left[\frac{e^{ik|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}}{4\pi|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\right] = -\delta^3(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'). \tag{51}$$

Hence it follows that the solution to eq. (49) is given by

$$g(r) = \frac{1}{4\pi}\int \frac{e^{ik|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,'|}\,h(r')d^3x'\,. \tag{52}$$

Indeed, plugging eq. (52) into the left-hand side of eq. (49) and employing eq. (51) produces the right-hand side of eq. (49) as required.

A computation similar to that of eq. (45) yields

$$\vec{\boldsymbol{\nabla}}^{\,2}\big[\vec{\boldsymbol{x}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}})\big] = -k^2\big[\vec{\boldsymbol{x}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}})\big]\,. \tag{53}$$

In light of eqs. (24) and (43), it follows that

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\vec{\boldsymbol{L}}^{\,2}\psi(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{L}}^{\,2}(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\psi(\vec{\boldsymbol{x}}) = 0\,. \tag{54}$$

Using eq. (18), we can conclude that $(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\psi(\vec{\boldsymbol{x}})$ is a radial function. A similar argument to the one given below eq. (48) implies that we can use the freedom to transform $\psi(\vec{\boldsymbol{x}}) \to \psi'(\vec{\boldsymbol{x}}) = \psi(\vec{\boldsymbol{x}}) + f(r)$ as indicated in eq. (9) to yield

$$(\vec{\boldsymbol{\nabla}}^{\,2} + k^2)\psi'(\vec{\boldsymbol{x}}) = 0\,. \tag{55}$$

Thus the second part of the Debye decomposition theorem indicated by eq. (10) is proven.