Electron wave function and mass renormalization in QED

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Abstract

In these notes, we compute the renormalized 1PI two-point Green function for electrons in QED at one loop order. Renormalization is carried out in the modified minimal subtraction scheme ($\overline{\rm MS}$) and the on-shell (OS) schemes, in a general covariant gauge using dimensional regularization. The wave function renormalization and mass renormalization constants, Z_2 and Z_m are explicitly evaluated. Special attention is given to the dependence on the gauge parameter. In the OS scheme, Z_2 exhibits an infrared divergence for (almost) all possible values of the gauge parameter, with one exception (corresponding to the Yennie gauge).

I. Introduction

The bare QED Lagrangian is given by

$$\mathscr{L}_{\rm QED} = -\tfrac{1}{4}F_B^{\mu\nu}F_{B\mu\nu} + \overline{\psi}_B(i\partial\!\!\!/ + e_B A\!\!\!/)\psi_B - m\overline{\psi}_B\psi_B - \frac{1}{2\xi_B}(\partial_\mu A_B^\mu)^2\,, \tag{1}$$

where the subscript B refers to bare parameters and fields. Introducing the renormalization constants to relate bare quantities to renormalized quantities (the latter with the B subscript removed), the renormalized parameters are defined via

$$e = Z_e^{-1}\mu^{-\epsilon}e_B\,, \qquad\qquad m = Z_m^{-1}m_B\,, \qquad\qquad \xi = Z_\xi^{-1}\xi_B\,, \tag{2}$$

where $\epsilon \equiv 2 - \frac{1}{2}n$ appears so that the renormalized coupling e is dimensionless when one-loop integrals are evaluated in n dimensions using dimensional regularization. Likewise, the renormalized fields are defined via

$$\psi = Z_2^{-1/2}\psi_B\,, \qquad\qquad A^\mu = Z_3^{-1/2}A_B^\mu\,. \tag{3}$$

It is traditional to introduce the vertex renormalization constant via

$$e\,\overline{\psi}A\!\!\!/\psi = \mu^{-\epsilon}Z_1^{-1}e_B\overline{\psi}_B A\!\!\!/\psi_B\,,$$

in which case we identify $Z_e = Z_1 Z_2^{-1}Z_3^{-1/2}$. One can also prove that

$$\frac{1}{2\xi}(\partial_\mu A^\mu)^2 = \frac{1}{2\xi_B}(\partial_\mu A_B^\mu)^2\,, \tag{4}$$

as a consequence of the Ward identities, which implies that $Z_\xi = Z_3$.

Inserting eqs. (2) and (3) into eq. (1) yields

$$\mathscr{L}_{\rm QED} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ + \mu^\epsilon e A\!\!\!/)\psi - m\overline{\psi}\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \mathcal{L}_{\rm CT}\,,$$

where the counterterm Lagrangian is given by

$$\mathcal{L}_{\rm CT} = -(Z_3-1)\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (Z_2-1)i\overline{\psi}\partial\!\!\!/\psi - (Z_m Z_2 - 1)m\overline{\psi}\psi + (Z_1-1)\mu^\epsilon e\overline{\psi}A\!\!\!/\psi\,.$$

Note that the counterterm Lagrangian does not contain a term proportional to the gauge parameter ξ in light of eq. (4). The counterterm Lagrangian is treated as a perturbation, which introduces additional Feynman rules for QED Green functions.

Working to one-loop order, it is convenient to define

$$\delta Z_i \equiv Z_i - 1\,, \quad (\text{for } i = 1,2,3)\,, \qquad \delta Z_m \equiv Z_m - 1\,.$$

At one loop, $\delta Z_i\,,\,\delta Z_m \sim \mathcal{O}(\alpha)$, where $\alpha \equiv e^2/(4\pi)$. Hence we can rewrite the counterterm Lagrangian at one-loop as

$$\mathcal{L}_{\rm CT} = -\tfrac{1}{4}\delta Z_3 F^{\mu\nu}F_{\mu\nu} + i\delta Z_2\overline{\psi}\partial\!\!\!/\psi - (\delta Z_m + \delta Z_2)m\overline{\psi}\psi + \delta Z_1\mu^\epsilon e\overline{\psi}A\!\!\!/\psi\,.$$

II. The 1PI electron two-point function

We now turn to the 1PI electron two-point function in momentum space,

$$i\Gamma^{(2)}(p) = i(p\!\!\!/ - m) - i\Sigma(p)\,, \tag{5}$$

where p is the four-momentum of the electron. Here, we have denoted the sum of the loop contributions to $i\Gamma^{(2)}(p)$ by $-i\Sigma(p)$. At one-loop the two contributing Feynman graphs are

where the cross indicates the contribution of the terms $i\delta Z_2\overline{\psi}\partial\!\!\!/\psi - (\delta Z_m + \delta Z_2)m\overline{\psi}\psi$ of the counterterm Lagrangian. Thus, at one loop,

$$-i\Sigma(p) = (i\mu^\epsilon e)^2\int\frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(q\!\!\!/ + p\!\!\!/ + m)\gamma^\mu}{q^2\big[(q+p)^2 - m^2\big]}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right) + i\delta Z_2 p\!\!\!/ - im(\delta Z_m + \delta Z_2)\,.$$

Using Dirac algebra in $n = 4 - 2\epsilon$ dimensions,

$$\gamma^\mu(q\!\!\!/ + p\!\!\!/ + m)\gamma_\mu = 2(\epsilon - 1)(q\!\!\!/ + p\!\!\!/) + (4 - 2\epsilon)m\,,$$

$$q\!\!\!/ p\!\!\!/ q\!\!\!/ = (2q\!\cdot\! p - p\!\!\!/ q\!\!\!/)q\!\!\!/ = 2q\!\cdot\! p\, q\!\!\!/ - q^2 p\!\!\!/\,,$$

it follow that

$$\int \frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(\not{q}+\not{p}+m)\gamma^\mu}{q^2\big[(q+p)^2-m^2\big]}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right)$$

$$= \int \frac{d^nq}{(2\pi)^n}\frac{2(\epsilon-1)(\not{q}+\not{p})+(4-2\epsilon)m}{q^2\big[(q+p)^2-m^2\big]} - (1-\xi)\int \frac{d^nq}{(2\pi)^n}\frac{q^2(\not{q}-\not{p}+m)+2q\cdot p\,\not{q}}{q^4\big[(q+p)^2-m^2\big]}\,.$$

$$= \int \frac{d^nq}{(2\pi)^n}\frac{2(\epsilon-1)(\not{q}+\not{p})+(4-2\epsilon)m-(1-\xi)(\not{q}-\not{p}+m)}{q^2\big[(q+p)^2-m^2\big]}$$

$$-2(1-\xi)p_\mu\gamma_\nu\int \frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{q^4\big[(q+p)^2-m^2\big]}\,. \tag{6}$$

Introducing Feynman parameters,

$$\int \frac{d^nq}{(2\pi)^n}\frac{1}{q^2\big[(q+p)^2-m^2\big]} = \int_0^1 dx\int \frac{d^nq}{(2\pi)^n}\frac{1}{\big[q^2+2q\cdot p\,x+x(p^2-m^2)\big]^2}$$

$$= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\int_0^1 dx\,x^{-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon}\,, \tag{7}$$

$$\int \frac{d^nq}{(2\pi)^n}\frac{q^\mu}{q^2\big[(q+p)^2-m^2\big]} = \int_0^1 dx\int \frac{d^nq}{(2\pi)^n}\frac{q^\mu}{\big[q^2+2q\cdot p\,x+x(p^2-m^2)\big]^2}$$

$$= -i(4\pi)^{\epsilon-2}\Gamma(\epsilon)p^\mu\int_0^1 dx\,x^{1-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon}\,, \tag{8}$$

$$\int \frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{q^4\big[(q+p)^2-m^2\big]} = 2\int_0^1 (1-x)\,dx\int \frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{\big[q^2+2q\cdot p\,x+x(p^2-m^2)\big]^3} \tag{9}$$

$$= -i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\int_0^1 dx\,(1-x)x^{-1-\epsilon}\big[m^2-p^2(1-x)\big]^{-1-\epsilon}\Big\{\epsilon x^2p^\mu p^\nu-\tfrac{1}{2}g^{\mu\nu}x\big[m^2-p^2(1-x)\big]\Big\}\,.$$

It then follows that

$$\int \frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(\not{q}+\not{p}+m)\gamma^\mu}{q^2\big[(q+p)^2-m^2\big]}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right)$$

$$= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\Bigg\{\Big[(3+\xi-2\epsilon)m-(1+\xi-2\epsilon)\not{p}\Big]\int_0^1 dx\,x^{-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon}$$

$$+(3-\xi-2\epsilon)\not{p}\int_0^1 dx\,x^{1-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon}$$

$$+2(1-\xi)\not{p}\int_0^1 dx\,(1-x)x^{-1-\epsilon}\big[m^2-p^2(1-x)\big]^{-1-\epsilon}\Big(\epsilon x^2p^2-\tfrac{1}{2}x\big[m^2-p^2(1-x)\big]\Big)\Bigg\}\,.$$

Hence, we end up with

$$\Sigma(p) = -\not{p} A(p^2) + m B(p^2)\,, \tag{10}$$

where

$$A(p^2) = \delta Z_2 + \frac{\alpha}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\int_0^1 dx\, x^{-\epsilon}\left[\frac{m^2 - p^2(1-x)}{\mu^2}\right]^{-\epsilon}$$
$$\times\left\{(1-x)(1-\epsilon) - x(1-\xi) - \frac{\epsilon(1-\xi)x(1-x)p^2}{m^2 - p^2(1-x)}\right\} \tag{11}$$

$$B(p^2) = \delta Z_m + \delta Z_2 + \frac{\alpha}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\big[\tfrac{1}{2}(3+\xi) - \epsilon\big]\int_0^1 dx\, x^{-\epsilon}\left[\frac{m^2 - p^2(1-x)}{\mu^2}\right]^{-\epsilon}, \tag{12}$$

after putting $e^2 = 4\pi\alpha$.

We recognize the ultraviolet divergence due to the presence of $\Gamma(\epsilon)$. But, if we attempt to take the on-shell limit, $p^2 = 0$, we obtain an indeterminant quantity in the $\epsilon \to 0$ limit.

III. $A(p^2)$ and $B(p^2)$ in terms of Passarino-Veltman loop functions

We can rewrite eqs. (7)–(9) in terms of Passarino-Veltman loop functions,

$$\int \frac{d^n q}{(2\pi)^n}\frac{1}{q^2\big[(q+p)^2 - m^2\big]} = \frac{i\mu^{-2\epsilon}}{16\pi^2}\, B_0(p^2; 0, m^2)\,, \tag{13}$$

$$\int \frac{d^n q}{(2\pi)^n}\frac{q^\mu}{q^2\big[(q+p)^2 - m^2\big]} = \frac{i\mu^{-2\epsilon}}{16\pi^2}\, p^\mu B_1(p^2; 0, m^2)\,, \tag{14}$$

$$\int \frac{d^n q}{(2\pi)^n}\frac{q^\mu q^\nu}{q^4\big[(q+p)^2 - m^2\big]} = \frac{i\mu^{-2\epsilon}}{16\pi^2}\big[p^\mu p^\nu C_{22}(0,p^2,p^2;0,0,m^2) + g^{\mu\nu} C_{24}(0,p^2,p^2;0,0,m^2)\big]\,. \tag{15}$$

It then follows that

$$\int \frac{d^n q}{(2\pi)^n}\frac{\gamma^\nu(\not{q} + \not{p} + m)\gamma^\mu}{q^2\big[(q+p)^2 - m^2\big]}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)$$
$$= \frac{i\mu^{-2\epsilon}}{16\pi^2}\Big\{\big[(2\epsilon - \xi - 1)\not{p} + (3 - 2\epsilon + \xi)m\big]B_0(p^2;0,m^2) + (2\epsilon + \xi - 3)\not{p} B_1(p^2;0,m^2)$$
$$-2(1-\xi)\not{p}\big[p^2 C_{22}(0,p^2,p^2;0,0,m^2) + C_{24}(0,p^2,p^2;0,0,m^2)\big]\Big\}\,. \tag{16}$$

Hence,

$$A(p^2) = \delta Z_2 + \frac{\alpha}{2\pi}\Big\{\big[\tfrac{1}{2}(1+\xi) - \epsilon\big]B_0(p^2;0,m^2) + \big[\tfrac{1}{2}(3-\xi) - \epsilon\big]B_1(p^2;0,m^2)$$
$$+(1-\xi)\big[p^2 C_{22}(0,p^2,p^2;0,0,m^2) + C_{24}(0,p^2,p^2;0,0,m^2)\big]\Big\}, \tag{17}$$

$$B(p^2) = \delta Z_m + \delta Z_2 + \frac{\alpha}{2\pi}\big[\tfrac{1}{2}(3+\xi) - \epsilon\big]B_0(p^2;0,m^2)\,. \tag{18}$$

Using the integral representations of the Passarino-Veltman loop functions, one can easily derive,

$$B_0(p^2;0,m^2) = (4\pi)^\epsilon\Gamma(\epsilon)\int_0^1 dx\, x^{-\epsilon}\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-\epsilon}, \tag{19}$$

$$B_1(p^2;0,m^2) = -(4\pi)^\epsilon\Gamma(\epsilon)\int_0^1 dx\, x^{1-\epsilon}\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-\epsilon}, \tag{20}$$

$$p^2C_{22}(0,p^2,p^2;0,0,m^2) = -(4\pi)^\epsilon\epsilon\Gamma(\epsilon)\int_0^1 dx\,\frac{p^2x(1-x)}{m^2-p^2(1-x)}x^{-\epsilon}\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-\epsilon}, \tag{21}$$

$$C_{24}(0,p^2,p^2;0,0,m^2) = \tfrac{1}{2}(4\pi)^\epsilon\Gamma(\epsilon)\int_0^1 dx\,(1-x)x^{-\epsilon}\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-\epsilon}, \tag{22}$$

Plugging these results into eqs. (17) and (18), we recover the results of eqs. (11) and (12).

IV. The limit of a zero mass electron

The limit of $m = 0$ is a subtle one. In this case, $\Sigma(p) = -\not{p}\big[\delta Z_2 + A_0(p^2)\big]$, where

$$A_0(p^2) = \frac{\alpha}{2\pi}(4\pi)^\epsilon\,(1-\epsilon)\,\Gamma(\epsilon)\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon}\int_0^1 x^{-\epsilon}(1-x)^{-\epsilon}(1-2x+\xi x)\,dx\,, \tag{23}$$

after employing the result of eq. (11). Noting the symmetry of the integrand under $x \to 1-x$, it follows that

$$\begin{aligned} A_0(p^2) &= \frac{\alpha\,\xi}{2\pi}(4\pi)^\epsilon\,(1-\epsilon)\,\Gamma(\epsilon)\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon}\int_0^1 x^{1-\epsilon}(1-x)^{-\epsilon}\,dx \\ &= \frac{\alpha\,\xi}{2\pi}(4\pi)^\epsilon\,\frac{\Gamma(\epsilon)\,\Gamma^2(2-\epsilon)}{\Gamma(3-2\epsilon)}\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon}. \end{aligned} \tag{24}$$

Recalling that $\epsilon = 2 - \frac{1}{2}n$, where n is the number of spacetime dimensions, we recognize the ultraviolet divergence due to the presence of $\Gamma(\epsilon)$. In particular, one must assume that $n < 4$ or equivalently $\epsilon > 0$, prior to the analytic continuation to $\epsilon = 0$. But, if we attempt to take the on-shell limit, $p^2 = 0$, we obtain an indeterminate quantity in the $\epsilon \to 0$ limit. One suggested strategy for dealing with this ambiguity is outlined on pp. 118–119 of Ref. [1].

However, a more direct approach can be adopted by setting $p^2 = m^2 = 0$ in eq. (6). It then follows that,

$$\begin{aligned} &\int \frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(\not{q}+\not{p})\gamma^\mu}{q^2(q+p)^2}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right) \\ &\quad = \int\frac{d^nq}{(2\pi)^n}\frac{2(\epsilon-1)(\not{q}+\not{p})-(1-\xi)(\not{q}-\not{p})}{q^2(q+p)^2} - 2(1-\xi)p_\mu\gamma_\nu\int\frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{q^4(q+p)^2}\,. \end{aligned} \tag{25}$$

If we write $(q+p)^2 = q^2 + 2q\cdot p$, then

$$\frac{1}{q^2(q^2-2q\cdot p)} = \int_0^1 \frac{dx}{\left[q^2+2xq\cdot p\right]^2}\,, \qquad \frac{1}{q^4(q^2-2q\cdot p)} = 2\int_0^1 \frac{(1-x)\,dx}{\left[q^2+2xq\cdot p\right]^3}\,. \tag{26}$$

Defining a new integration variable, $Q = q + xp$, it follows that $Q^2 = q^2 + 2xq\cdot p$ and

$$\begin{aligned}\int \frac{d^nq}{(2\pi)^n}\frac{2(\epsilon-1)(\not{q}+\not{p})-(1-\xi)(\not{q}-\not{p})}{q^2(q+p)^2} &= \not{p}\int_0^1 \left[2(\epsilon-1)(1-x)+(1-\xi)(1+x)\right]dx \int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4} \\ &= \not{p}\left[\tfrac{1}{2}+\epsilon-\tfrac{3}{2}\xi\right]\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,. \end{aligned}\tag{27}$$

and

$$p_\mu\gamma_\nu \int \frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{q^4(q+p)^2} = 2p_\mu\gamma_\nu\int_0^1(1-x)\,dx\int\frac{d^nQ}{(2\pi)^n}\frac{Q^\mu Q^\nu}{Q^6} = \frac{1}{n}\not{p}\int\frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,. \tag{28}$$

Hence, after putting $n = 4-2\epsilon$,

$$\int \frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(\not{q}+\not{p})\gamma^\mu}{q^2(q+p)^2}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right) = \not{p}\left[\frac{1}{2}+\epsilon-\frac{3\xi}{2}-\frac{1-\xi}{2-\epsilon}\right]\int\frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,. \tag{29}$$

Strictly speaking, the integral

$$\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,, \tag{30}$$

is undefined for any value of n. In particular, it is both ultraviolet and infrared divergent for $n=4$. Following the conventions of dimensional regularization (see, e.g., Ref. [2]), one defines integrals with no explicit scale to be zero,

$$\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^p} = 0\,, \tag{31}$$

for any power p. In particular, one can understand the vanishing of eq. (30) in dimensional regularization as a consequence of an exact cancellation of the infrared and ultraviolet divergence.

In order to see this cancellation explicitly, we shall rewrite eq. (30) following eq. (C.22) of Ref. [3],

$$\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4} = \int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^2(Q^2-m^2)} - \int\frac{d^nQ}{(2\pi)^n}\frac{m^2}{Q^4(Q^2-m^2)}\,. \tag{32}$$

This result clearly exhibits the infrared and ultraviolet divergences, but relegates them to separate integrals. It then follows from eq. (7) that

$$\begin{aligned}\int\frac{d^nQ}{(2\pi)^n}\frac{1}{Q^2(Q^2-m^2)} &= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)(m^2)^{-\epsilon}\int_0^1 x^{-\epsilon}\,dx \\ &= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)+1-\ln m^2\right)+\mathcal{O}(\epsilon)\,,\end{aligned}\tag{33}$$

where $n = 4-2\epsilon$, which exhibits an ultraviolet divergence when the limit of $\epsilon\to 0$ is taken.

The second integral on the right-hand side of eq. (32) is infrared divergent. To distinguish this divergence from the ultraviolet divergence in eq. (33), we will write $n = 4 - 2\epsilon'$ in the following computation,

$$\begin{aligned}
\int \frac{d^nQ}{(2\pi)^n}\frac{m^2}{Q^4(Q^2-m^2)} &= 2m^2\int_0^1 x\,dx\int\frac{d^nq}{(2\pi)^n}\frac{1}{\big[q^2-(1-x)m^2)\big]^3} \\
&= -i(4\pi)^{\epsilon'-2}(m^2)^{-\epsilon'}\,\Gamma(1+\epsilon')\int_0^1 x(1-x)^{-1-\epsilon'}\,dx \\
&= -i(4\pi)^{\epsilon'-2}(m^2)^{-\epsilon'}\,\frac{\Gamma(1+\epsilon')\Gamma(-\epsilon')}{\Gamma(2-\epsilon')} = \frac{i}{(4\pi)^2}\left(\frac{4\pi}{m^2}\right)^{\epsilon'}\frac{\Gamma(1+\epsilon')}{\epsilon'(1-\epsilon')} \\
&= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon'}-\gamma+\ln(4\pi)+1-\ln m^2\right)+\mathcal{O}(\epsilon')\,. \qquad (34)
\end{aligned}$$

Hence, eq. (32) yields,

$$\int\frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon}-\frac{1}{\epsilon'}\right) = 0\,, \qquad (35)$$

after using $n = 4-2\epsilon = 4-2\epsilon'$, which demonstrates the exact cancellation of the infrared and ultraviolet divergences as asserted below eq. (31).

Of course, the statement that $A_0(p^2=0)$ does not imply that the divergence is absent. Indeed, δZ_2 can be unambiguously determined the $\overline{\text{MS}}$ scheme where no infrared divergences are present as we will show in Sections V. In the computation of physical observables at one-loop, the implication of $A(p^2=0)=0$ in massless QED is simply that one can neglect Feynman diagrams that contain self-energy corrections on the outgoing electrons and positron lines. The counterterms on the external legs of the diagram are still present, and will end up being reinterpreted as contributing to the infrared divergence, which will ultimately cancel infrared divergences arising from other Feynman graphs, since physical observables are necessarily infrared safe.

V. The renormalized 1PI electron two-point function in the $\overline{\text{MS}}$ scheme

If we use $\overline{\text{MS}}$ subtraction to fix the counterterms, then

$$\begin{aligned}
\delta Z_2^{\overline{\text{MS}}} &= -\frac{\alpha}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\int_0^1 dx\,(1-2x+x\xi) = -\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,, \\
\delta Z_m^{\overline{\text{MS}}}+\delta Z_2^{\overline{\text{MS}}} &= -\frac{\alpha(3+\xi)}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,.
\end{aligned}$$

Hence,

$$\delta Z_2^{\overline{\text{MS}}} = -\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,, \qquad \delta Z_m^{\overline{\text{MS}}} = -\frac{3\alpha}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,, \qquad (36)$$

where

$$(4\pi)^\epsilon\,\Gamma(\epsilon) = \frac{1}{\epsilon}-\gamma+\ln 4\pi+\mathcal{O}(\epsilon)\,.$$

Note that δZ_2 is gauge dependent, whereas δZ_m is gauge independent.

We can easily reproduce the results of eq. (36) using the Passarion-Veltman functions. In particular, the $\overline{\text{MS}}$ procedure instructs us to set $A(p^2) = B(p^2) = 0$ in eqs. (17) and (18) and identify

$$B_0(p^2;0,m^2) = (4\pi)^\epsilon\Gamma(\epsilon)\,, \qquad B_1(p^2;0,m^2) = -\tfrac{1}{2}(4\pi)^\epsilon\Gamma(\epsilon)\,, \tag{37}$$

$$C_{24}(0,p^2,p^2;0,0,m^2) = \tfrac{1}{4}(4\pi)^\epsilon\Gamma(\epsilon)\,, \qquad C_{22}(0,p^2,p^2;0,0,m^2) = 0\,. \tag{38}$$

Solving for δZ_2 and δZ_m, and setting $\epsilon = 0$ except in the prefactors, we recover eq. (36).

Inserting eq. (36) back into eqs. (11) and (12) and taking the $\epsilon \to 0$ limit, we obtain

$$A(p^2)_{\overline{\text{MS}}} = -\frac{\alpha}{2\pi}\left\{\int_0^1 \left[1 - x + \frac{x(1-x)(1-\xi)p^2}{m^2 - p^2(1-x)}\right] dx \right.$$
$$\left. + \int_0^1 (1 - 2x + x\xi)\left[\ln x + \ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right)\right] dx\right\},$$

$$B(p^2)_{\overline{\text{MS}}} = -\frac{\alpha}{2\pi}\left\{1 + \tfrac{1}{2}(3+\xi)\int_0^1 \left[\ln x + \ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right)\right] dx\right\},$$

The relevant integrals are

$$\int_0^1 \frac{x(1-x)dx}{m^2 - p^2(1-x)} = \frac{2m^2 - p^2}{2p^4} - \frac{m^2}{p^4}\left(1 - \frac{m^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\,,$$

$$\int_0^1 x^n \ln x\, dx = -\frac{1}{(n+1)^2}\,, \quad \text{for } n = 0,1,2,\ldots,$$

$$\int_0^1 \ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right) dx = \ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^2}{p^2}\ln\left(1 - \frac{p^2}{m^2}\right) - 1\,,$$

$$\int_0^1 (1-x)\ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right) dx = \frac{1}{2}\ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^4}{2p^4}\ln\left(1 - \frac{p^2}{m^2}\right) - \frac{m^2}{2p^2} - \frac{1}{4}\,.$$

It follows that

$$A(p^2)_{\overline{\text{MS}}} = \frac{\alpha\,\xi}{4\pi}\left[1 + \frac{m^2}{p^2} - \ln\left(\frac{m^2 - p^2}{\mu^2}\right) + \frac{m^4}{p^4}\ln\left(1 - \frac{p^2}{m^2}\right)\right]\,,$$

$$B(p^2)_{\overline{\text{MS}}} = \frac{\alpha}{2\pi}\left\{2 + \xi - \tfrac{1}{2}(3+\xi)\left[\ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^2}{p^2}\ln\left(1 - \frac{p^2}{m^2}\right)\right]\right\}\,.$$

Note that A and B are finite for $p^2 = m^2$,

$$A(m^2)_{\overline{\text{MS}}} = \frac{\alpha\,\xi}{4\pi}\left[2 - \ln\left(\frac{m^2}{\mu^2}\right)\right]\,, \qquad B(m^2)_{\overline{\text{MS}}} = \frac{\alpha}{2\pi}\left[2 + \xi - \tfrac{1}{2}(3+\xi)\ln\left(\frac{m^2}{\mu^2}\right)\right]\,. \tag{39}$$

In light of eqs. (5) and (10), the one-loop correction to the inverse propagator is

$$\begin{aligned}
\Gamma^{(2)}(p)_{\overline{\rm MS}} &= \not p - m - \Sigma(p)_{\overline{\rm MS}} = \not p\big[1 + A(p^2)_{\overline{\rm MS}}\big] - m\big[1 + B(p^2)_{\overline{\rm MS}}\big] \\
&= \not p\left\{1 + \frac{\alpha\,\xi}{4\pi}\left[1 + \frac{m^2}{p^2} - \ln\left(\frac{m^2 - p^2}{\mu^2}\right) + \frac{m^4}{p^4}\ln\left(1 - \frac{p^2}{m^2}\right)\right]\right\} \\
&\quad -m\left\{1 + \frac{\alpha}{2\pi}\left(2 + \xi - \tfrac{1}{2}(3+\xi)\left[\ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^2}{p^2}\ln\left(1 - \frac{p^2}{m^2}\right)\right]\right)\right\}. \qquad (40)
\end{aligned}$$

In eq. (40), $m \equiv m(\mu)$ is the renormalized mass, which differs from the physical pole mass. The definition of the $\overline{\rm MS}$ mass is obtained by setting $\mu = m$. That is, the $\overline{\rm MS}$ mass is defined as $m_R \equiv m(m)$. Thus, we set $\mu = m$ in eq. (40) and obtain,

$$\begin{aligned}
\Gamma^{(2)}(p)_{\overline{\rm MS}} &= \not p\left\{1 + \frac{\alpha\,\xi}{4\pi}\left[1 + \frac{m_R^2}{p^2} - \left(1 - \frac{m_R^4}{p^4}\right)\ln\left(1 - \frac{p^2}{m_R^2}\right)\right]\right\} \\
&\quad -m_R\left\{1 + \frac{\alpha}{2\pi}\left[2 + \xi - \tfrac{1}{2}(3+\xi)\left(1 - \frac{m_R^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m_R^2}\right)\right]\right\}. \qquad (41)
\end{aligned}$$

The physical pole mass, denoted by m_e, corresponds to a zero of the inverse propagator. That is, m_e is defined by the condition

$$\Gamma^{(2)}(p)\Big|_{\not p = m_e} = 0\,. \qquad (42)$$

The simplest way to obtain an expression for m_e at one loop accuracy is to rewrite eq. (40) as

$$\Gamma^{(2)}(p)_{\overline{\rm MS}} = \big[1 + A(p^2)_{\overline{\rm MS}}\big]\left[\not p - m\left(\frac{1 + B(p^2)_{\overline{\rm MS}}}{1 + A(p^2)_{\overline{\rm MS}}}\right)\right].$$

Since $A(p^2)_{\overline{\rm MS}}$ and $B(p^2)_{\overline{\rm MS}}$ are quantities of $\mathcal{O}(\alpha)$, then to one-loop accuracy,

$$\Gamma^{(2)}(p)_{\overline{\rm MS}} \simeq \big[1 + A(p^2)_{\overline{\rm MS}}\big]\big[\not p - m\big(1 + B(p^2)_{\overline{\rm MS}} - A(p^2)_{\overline{\rm MS}}\big)\big].$$

We can then immediately identify

$$m_e = m\big[1 + B(m_e^2)_{\overline{\rm MS}} - A(m_e^2)_{\overline{\rm MS}}\big].$$

At one-loop accuracy, $A(m_e^2) = A(m^2)\big|_{\mu=m}$ and $B(m_e^2) = B(m^2)\big|_{\mu=m}$. Hence, using eq. (39), we end up with

$$m_e = m_R\left(1 + \frac{\alpha}{\pi}\right).$$

Although the quantity $B(p^2)_{\overline{\rm MS}} - A(p^2)_{\overline{\rm MS}}$ is gauge-invariant on-shell, it depends on the gauge parameter ξ off-shell. In particular,

$$B(p^2)_{\overline{\rm MS}} - A(p^2)_{\overline{\rm MS}} = \frac{\alpha}{4\pi}\left\{4 + \xi\left(1 - \frac{m^2}{p^2}\right) - 3\ln\left(\frac{m^2 - p^2}{\mu^2}\right) + \frac{m^2}{p^2}\left[3 + \xi\left(1 - \frac{m^2}{p^2}\right)\right]\ln\left(1 - \frac{p^2}{m^2}\right)\right\}.$$

One can easily check that $B(m^2) - A(m^2) = \alpha/\pi$ as required.

Finally, we note that the $\overline{\rm MS}$ scheme is a mass-independent scheme. In particular, no infrared divergences appear in the evaluation of the 1PI electron two-point function in this scheme.

VI. The renormalized 1PI electron two-point function in the on-shell (OS) scheme

Consider the on-shell (OS) renormalization scheme, where we identify the parameter m as the pole mass. In this case, we expand,

$$\Sigma(p) = \Sigma(m) + (\not p - m)\Sigma'(m) + \mathcal{O}\big((\not p - m)^2\big)\,.$$

The renormalization conditions that ensure that the residue of the propagator is unity and the pole of the propagator is the pole mass m are:

$$\Sigma(m)_{\rm OS} = 0\,, \qquad\qquad \Sigma'(m)_{\rm OS} = 0\,. \tag{43}$$

It then follows that the inverse propagator can be written as

$$\begin{aligned}\Gamma^{(2)}(p)_{\rm OS} = \not p - m - \Sigma(p)_{\rm OS} &= \big[1+\Sigma'(m)_{\rm OS}\big](\not p - m) - \Sigma(m)_{\rm OS} + \mathcal{O}\big((\not p - m)^2\big)\\ &= \not p - m + \mathcal{O}\big((\not p - m)^2\big)\,.\end{aligned}$$

Employing eq. (10), we can rewrite the boundary conditions specified in eq. (43) as

$$A(m^2)_{\rm OS} = B(m^2)_{\rm OS}\,, \qquad\qquad A(m^2)_{\rm OS} = 2m^2\left[\left(\frac{\partial B_{\rm OS}}{\partial p^2}\right) - \left(\frac{\partial A_{\rm OS}}{\partial p^2}\right)\right]_{p^2=m^2}\,, \tag{44}$$

where we have used $\not p\not p = p^2$ and

$$\frac{\partial}{\partial\not p} = 2\not p\frac{\partial}{\partial p^2}\,.$$

Using eqs. (11) and (12),

$$A(m^2)_{\rm OS} = \delta Z_2 + \frac{\alpha}{2\pi}(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\int_0^1 dx\, x^{-2\epsilon}\big[(1-x)\big(1-(2-\xi)\epsilon\big) - x(1-\xi)\big]$$

$$B(m^2)_{\rm OS} = \delta Z_m + \delta Z_2 + \frac{\alpha}{2\pi}\left(\tfrac{1}{2}(3+\xi) - \epsilon\right)(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\int_0^1 dx\, x^{-2\epsilon}\,.$$

The integrals above are elementary; the end result is

$$A(m^2)_{\rm OS} = \delta Z_2^{\rm OS} + \frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{1}{1-2\epsilon}\,, \tag{45}$$

$$B(m^2)_{\rm OS} = \delta Z_m^{\rm OS} + \delta Z_2^{\rm OS} + \frac{\alpha}{2\pi}\left(\tfrac{1}{2}(3+\xi) - \epsilon\right)(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{1}{1-2\epsilon}\,. \tag{46}$$

Using eq. (44), we conclude that

$$\delta Z_m^{\rm OS} = -\frac{\alpha}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\left(\frac{3-2\epsilon}{1-2\epsilon}\right) = -\frac{3\alpha}{4\pi}\left[(4\pi)^\epsilon\Gamma(\epsilon)+\tfrac{4}{3}-\ln\left(\frac{m^2}{\mu^2}\right)\right], \qquad (47)$$

after dropping terms of $\mathcal{O}(\epsilon)$.

Next, we compute derivatives of eqs. (11) and (12) with respect to p^2,

$$\frac{\partial A_{\rm OS}}{\partial p^2} = \frac{\alpha}{2\pi\mu^2}(4\pi)^\epsilon\Gamma(1+\epsilon)\int_0^1 dx\, x^{-\epsilon}(1-x)\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-1-\epsilon}$$
$$\times\left\{(1-x)(1-\epsilon)-2x(1-\xi)-\frac{(1-\xi)(1+\epsilon)x(1-x)p^2}{m^2-p^2(1-x)}\right\}, \qquad (48)$$

$$\frac{\partial B_{\rm OS}}{\partial p^2} = \frac{\alpha}{2\pi\mu^2}(4\pi)^\epsilon\Gamma(1+\epsilon)\left[\tfrac{1}{2}(3+\xi)-\epsilon\right]\int_0^1 dx\, x^{-\epsilon}(1-x)\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-1-\epsilon}. \qquad (49)$$

Note that these quantities are ultraviolet finite, after using $\epsilon\Gamma(\epsilon)=\Gamma(1+\epsilon)$. If we now set $p^2=m^2$, we see that the integrands in eqs. (48) and (49) behave as $x^{-1+2\epsilon}$ as $x\to 0$. Thus integrating over x generates infrared divergences, which are regulated when $\epsilon\neq 0$. The resulting integrals are elementary, and it follows that

$$\left(\frac{\partial A_{\rm OS}}{\partial p^2}\right)_{p^2=m^2} = -\frac{\alpha\,\xi}{4\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}, \qquad (50)$$

$$\left(\frac{\partial B_{\rm OS}}{\partial p^2}\right)_{p^2=m^2} = -\frac{\alpha}{4\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\left[\tfrac{1}{2}(3+\xi)-\epsilon\right]. \qquad (51)$$

In light of eqs. (44) and (45),

$$\delta Z_2^{\rm OS} = -\frac{\alpha\,\xi}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(\epsilon)}{1-2\epsilon}+\frac{\alpha}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\left[\xi-3+2\epsilon\right]. \qquad (52)$$

The term on the right hand side of eq. (52) proportional to $\Gamma(\epsilon)$ represents the ultraviolet divergence [cf. eq. (36)]. The last term on the right hand side of eq. (52) which contains a pole at $\epsilon=0$ corresponds to the infrared divergence. Note that the infrared divergence at one loop is absent in the Yennie gauge, which corresponds to $\xi=3$.

We can add the two terms on the right hand side of eq. (52), if we are not concerned about the mixing of the infrared and the ultraviolet divergences. The end result is

$$\delta Z_2^{\rm OS} = -\frac{\alpha}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(3-2\epsilon)(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}. \qquad (53)$$

The ultraviolet divergence cancels part of the infrared divergence. Remarkably, the end result is independent of the gauge parameter ξ.

Using eqs. (11) and (12), we can determine $A(p^2)$ and $B(p^2)$ in the on-shell scheme by writing

$$A(p^2)_{\rm OS} = A(p^2)_{\overline{\rm MS}}+\delta Z_2^{\rm OS}-\delta Z_2^{\overline{\rm MS}},$$
$$B(p^2)_{\rm OS} = B(p^2)_{\overline{\rm MS}}+\delta Z_m^{\rm OS}+\delta Z_2^{\rm OS}-\delta Z_m^{\overline{\rm MS}}-\delta Z_2^{\overline{\rm MS}}.$$

Eqs. (36), (47) and (52) yield,

$$\delta Z_2^{\rm OS} - \delta Z_2^{\overline{\rm MS}} = -\frac{\alpha\,\xi}{2\pi}\left[1-\tfrac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right] + \frac{\alpha}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\bigl[\xi-3+2\epsilon\bigr]\,. \quad (54)$$

$$\delta Z_m^{\rm OS} - \delta Z_m^{\overline{\rm MS}} = -\frac{\alpha}{\pi}\left[1-\tfrac{3}{4}\ln\left(\frac{m^2}{\mu^2}\right)\right]\,. \quad (55)$$

The infrared divergence is explicitly exhibited in eq. (54). Expanding about $\epsilon = 0$ yields

$$\delta Z_2^{\rm OS} - \delta Z_2^{\overline{\rm MS}} = \frac{\alpha(\xi-3)}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon) - \frac{\alpha}{\pi}\left[1-\tfrac{3}{4}\ln\left(\frac{m^2}{\mu^2}\right)\right]\,. \quad (56)$$

Thus, both $A(p^2)_{\rm OS}$ and $B(p^2)_{\rm OS}$ are infrared divergent if $\xi \neq 3$. However the difference $B(p^2)_{\rm OS} - A(p^2)_{\rm OS}$ is infrared finite, although it depends on the gauge parameter ξ for $p^2 \neq m^2$.

It is instructive to check the results obtained in this section by employing eqs. (17) and (18). Then, eq. (44) yields,

$$\begin{aligned}\delta Z_m^{\rm OS} = -\frac{\alpha}{2\pi}\biggl\{&\bigl[B_0(m^2;0,m^2) - \bigl[\tfrac{1}{2}(3-\xi)-\epsilon\bigr]B_1(m^2;0,m^2)\,,\\ &-(1-\xi)\bigl[m^2C_{22}(0,m^2,m^2;0,0,m^2)+C_{24}(0,m^2,m^2;0,0,m^2)\bigr]\biggr\}.\end{aligned} \quad (57)$$

Using eqs. (20)–(22), it follows that

$$m^2C_{22}(0,m^2,m^2;0,0,m^2)+C_{24}(0,m^2,m^2;0,0,m^2) = -\tfrac{1}{2}B_1(m^2;0,m^2) = \frac{(4\pi)^\epsilon\Gamma(\epsilon)}{4(1-\epsilon)}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}. \quad (58)$$

Hence, it follows that

$$\delta Z_m^{\rm OS} = -\frac{\alpha}{2\pi}\bigl[B_0(m^2;0,m^2) - (1-\epsilon)B_1(m^2;0,m^2)\bigr] \quad (59)$$

In light of eq. (19),

$$B_0(m^2;0,m^2) = \frac{(4\pi)^\epsilon\Gamma(\epsilon)}{1-2\epsilon}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\,. \quad (60)$$

Employing the results of eqs. (58) and (60) in eq. (59), we recover the result of eq. (47).

Likewise, eq. (44) yields,

$$\begin{aligned}\delta Z_2^{\rm OS} = -\frac{\alpha}{2\pi}\biggl\{&\bigl[\tfrac{1}{2}(1+\xi)-\epsilon\bigr]B_0(m^2;0,m^2) + \bigl[\tfrac{1}{2}(3-\xi)-\epsilon\bigr]B_1(m^2;0,m^2)\\ &+(1-\xi)\bigl[m^2C_{22}(0,m^2,m^2;0,0,m^2)+C_{24}(0,m^2,m^2;0,0,m^2)\bigr]\\ &-2m^2B_0'(m^2;0,m^2) + 2m^2\bigl[\tfrac{1}{2}(3-\xi)-\epsilon\bigr]B_1'(m^2;0,m^2)\\ &+2m^2(1-\xi)\bigl[C_{22}(0,m^2,m^2;0,0,m^2)+m^2C_{22}'(0,m^2,m^2;0,0,m^2)\\ &+C_{24}'(0,m^2,m^2;0,0,m^2)\bigr]\biggr\}\,,\end{aligned} \quad (61)$$

where the prime indicates

$$B_i'(m^2;0,m^2) \equiv \frac{\partial}{\partial p^2}\, B_i(p^2;0,m^2)\bigg|_{p^2=m^2}\,, \tag{62}$$

$$C_{ij}'(0,m^2,m^2;0,0,m^2) \equiv \frac{\partial}{\partial p^2}\, C_{ij}(0,p^2,p^2;0,0,m^2)\bigg|_{p^2=m^2}\,. \tag{63}$$

Using eq. (58), the expression for $\delta Z_2^{\rm OS}$ can be rewritten as,

$$\begin{aligned}\delta Z_2^{\rm OS} = -\frac{\alpha}{2\pi}\bigg\{ & \left[\tfrac{1}{2}(1+\xi)-\epsilon\right] B_0(m^2;0,m^2) + (1-\epsilon) B_1(m^2;0,m^2) \\ & -2m^2 B_0'(m^2;0,m^2) + 2m^2\left[\tfrac{1}{2}(3-\xi)-\epsilon\right] B_1'(m^2;0,m^2) \\ & +2m^2(1-\xi)\big[C_{22}(0,m^2,m^2;0,0,m^2) + m^2 C_{22}'(0,m^2,m^2;0,0,m^2) \\ & +C_{24}'(0,m^2,m^2;0,0,m^2)\big]\bigg\}\,,\end{aligned} \tag{64}$$

It is straightforward to obtain,

$$2m^2 B_0'(m^2;0,m^2) = -\frac{(4\pi)^\epsilon\Gamma(\epsilon)}{1-2\epsilon}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\,, \tag{65}$$

$$2m^2 B_1'(m^2;0,m^2) = -\frac{(4\pi)^\epsilon\epsilon\Gamma(\epsilon)}{(1-\epsilon)(1-2\epsilon)}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\,, \tag{66}$$

$$2m^2 C_{22}(0,m^2,m^2;0,0,m^2) = -\frac{(4\pi)^\epsilon\epsilon\Gamma(\epsilon)}{(1-\epsilon)(1-2\epsilon)}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\,, \tag{67}$$

$$2m^4 C_{22}'(0,m^2,m^2;0,0,m^2) = \frac{(4\pi)^\epsilon(1+\epsilon)\Gamma(\epsilon)}{(1-\epsilon)(1-2\epsilon)}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\,, \tag{68}$$

$$2m^2 C_{24}'(0,m^2,m^2;0,0,m^2) = -\frac{(4\pi)^\epsilon\Gamma(\epsilon)}{2(1-\epsilon)(1-2\epsilon)}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\,. \tag{69}$$

All divergences that appear above are infrared divergences. We can simplify our expression for $\delta Z_2^{\rm OS}$ by employing,

$$2m^2\big[C_{22}(0,m^2,m^2;0,0,m^2)+m^2C_{22}'(0,m^2,m^2;0,0,m^2)+C_{24}'(0,m^2,m^2;0,0,m^2)\big] = -\frac{1}{2\epsilon}B_1'(m^2;0,m^2)\,. \tag{70}$$

We then end up with

$$\begin{aligned}\delta Z_2^{\rm OS} = -\frac{\alpha}{2\pi}\bigg\{ & \left[\tfrac{1}{2}(1+\xi)-\epsilon\right] B_0(m^2;0,m^2) + (1-\epsilon) B_1(m^2;0,m^2) \\ & -2m^2 B_0'(m^2;0,m^2) - 2m^2\left[\frac{(1-\epsilon)(1-\xi-2\epsilon)}{2\epsilon}\right] B_1'(m^2;0,m^2)\bigg\}\,,\end{aligned} \tag{71}$$

which reproduces the result of eq. (52).

VII. The functions $A(p^2)$ and $B(p^2)$ in terms of hypergeometric functions

It is sometimes convenient to evaluate the functions $A(p^2)$ and $B(p^2)$ prior to taking the $\epsilon \to 0$ limit. In this case, $A(p^2)$ and $B(p^2)$ can be expressed in terms of hypergeometric functions. First, we define the following family of integrals:

$$I_{n,\ell}(p^2) \equiv \int_0^1 dx\, x^{n-\epsilon} \left[\frac{m^2 - p^2(1-x)}{m^2}\right]^{-\epsilon-\ell}, \tag{72}$$

Using the integral representation of the Gauss hypergeometric function,

$$F(a\,,\,b\,;\,c\,;\,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-xz)^{-a}\,dx\,,$$

for $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$ and $|\arg(1-z)| < \pi$, and the functional relation,

$$F(a\,,\,b\,;\,c\,;\,z) = (1-z)^{-a} F\left(a\,,\,c-b\,;\,c\,;\,\frac{z}{z-1}\right)\,,$$

it follows that

$$I_{n,\ell}(p^2) = \frac{1}{n+1-\epsilon}\, F\left(\ell+\epsilon\,,\,1\,;\,n+2-\epsilon\,;\,\frac{p^2}{m^2}\right)\,. \tag{73}$$

It is convenient to rewrite $I_{n,0}(\epsilon)$ in another form using the recursion relation,

$$(c-a-b)F(a\,,\,b\,;\,c\,;\,z) + a(1-z)F(a+1\,,\,b\,;\,c\,;\,z) - (c-b)F(a\,,\,b-1\,;\,c\,;\,z) = 0\,.$$

It follows that

$$I_{n,0}(p^2) = \frac{1}{n+1-2\epsilon}\left[1 - \frac{\epsilon}{n+1-\epsilon}\left(1-\frac{p^2}{m^2}\right) F\left(1+\epsilon\,,\,1\,;\,n+2-\epsilon\,;\,\frac{p^2}{m^2}\right)\right]\,. \tag{74}$$

Using eqs. (11) and (12),

$$A(p^2) = \delta Z_2 + \frac{\alpha}{2\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\Gamma(\epsilon)\Big\{(1-\epsilon)\left[I_{0,0}(p^2) - I_{1,0}(p^2)\right] - (1-\xi)I_{1,0}(\epsilon)$$
$$-\epsilon(1-\xi)\frac{p^2}{m^2}\left[I_{1,1}(p^2) - I_{2,1}(p^2)\right]\Big\} \tag{75}$$

$$B(p^2) = \delta Z_m + \delta Z_2 + \frac{\alpha}{2\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\Gamma(\epsilon)\left[\tfrac{1}{2}(3+\xi) - \epsilon\right] I_{0,0}(p^2)\,. \tag{76}$$

From eqs. (75) and (76), one can easily perform the expansion in ϵ. In particular,

$$I_{n,0}(p^2) = \frac{1}{n+1}\left[1 + \frac{2\epsilon}{n+1} - \frac{\epsilon}{n+1}\left(1-\frac{p^2}{m^2}\right) F\left(1\,,\,1\,;\,n+2\,;\,\frac{p^2}{m^2}\right)\right] + \mathcal{O}(\epsilon^2)\,,$$

$$I_{n,1}(p^2) = \frac{1}{n+1} F\left(1\,,\,1\,;\,n+2\,;\,\frac{p^2}{m^2}\right) + \mathcal{O}(\epsilon)\,,$$

where

$$F\left(1\,,\,1\,;\,n+2\,;\,z\right)=-\frac{(1+n)(1-z)^n}{n!}\frac{d^n}{dz^n}\left[\frac{\ln(1-z)}{z}\right]\,.$$

In particular,

$$F\left(1\,,\,1\,;\,2\,;\,z\right)=-\frac{\ln(1-z)}{z}\,, \tag{77}$$

$$F\left(1\,,\,1\,;\,3\,;\,z\right)=2\left[\frac{1}{z}+\frac{(1-z)\ln(1-z)}{z^2}\right]\,, \tag{78}$$

$$F\left(1\,,\,1\,;\,4\,;\,z\right)=-\frac{3}{2}\left[\frac{2-3z}{z^2}+\frac{2(1-z)^2\ln(1-z)}{z^3}\right]\,. \tag{79}$$

It follows that

$$\begin{aligned}A(p^2)=\delta Z_2+\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)+\frac{\alpha}{2\pi}\Bigg\{&\frac{3}{2}-\left(1-\frac{p^2}{m^2}\right)F\left(1\,,\,1\,;\,2\,;\,\frac{p^2}{m^2}\right)\\ &-(1-\tfrac{1}{2}\xi)\left[1-\frac{1}{2}\left(1-\frac{p^2}{m^2}\right)F\left(1\,,\,1\,;\,3\,;\,\frac{p^2}{m^2}\right)\right]\\ &-(1-\xi)\frac{p^2}{m^2}\left[\tfrac{1}{2}F\left(1\,,\,1\,;\,3\,;\,\frac{p^2}{m^2}\right)-\tfrac{1}{3}F\left(1\,,\,1\,;\,4\,;\,\frac{p^2}{m^2}\right)\right]-\tfrac{1}{2}\xi\ln\left(\frac{m^2}{\mu^2}\right)\Bigg\}+\mathcal{O}(\epsilon)\,.\end{aligned}$$

$$\begin{aligned}B(p^2)=\delta Z_m+\delta Z_2&+\frac{\alpha}{4\pi}(3+\xi)(4\pi)^\epsilon\,\Gamma(\epsilon)\\ &+\frac{\alpha}{2\pi}\Bigg\{2+\xi-\tfrac{1}{2}(3+\xi)\left[\left(1-\frac{p^2}{m^2}\right)F\left(1\,,\,1\,;\,2\,;\,\frac{p^2}{m^2}\right)+\ln\left(\frac{m^2}{\mu^2}\right)\right]\Bigg\}+\mathcal{O}(\epsilon)\,.\end{aligned}$$

Inserting the results of eqs. (77)–(79) yields

$$A(p^2)=\delta Z_2+\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)+\frac{\alpha\,\xi}{4\pi}\left\{\left(1+\frac{m^2}{p^2}\right)\left[1-\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)\right]-\ln\left(\frac{m^2}{\mu^2}\right)\right\}+\mathcal{O}(\epsilon)\,,$$

$$\begin{aligned}B(p^2)=\delta Z_m+\delta Z_2&+\frac{\alpha}{4\pi}(3+\xi)(4\pi)^\epsilon\,\Gamma(\epsilon)\\ &+\frac{\alpha}{2\pi}\left\{2+\xi-\tfrac{1}{2}(3+\xi)\left[\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)+\ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}+\mathcal{O}(\epsilon)\,,\end{aligned}$$

which are equivalent to the results previously obtained.

In the OS scheme, we also need to compute the derivatives of $A(p^2)$ and $B(p^2)$ with respect to p^2. It is straightforward to obtain,

$$\frac{\partial I_{n,0}}{\partial p^2}=\frac{\epsilon}{m^2(n+1-\epsilon)(n+2-\epsilon)}F\left(1+\epsilon\,,\,2\,;\,n+3-\epsilon\,;\,\frac{p^2}{m^2}\right)\,,$$

$$\frac{\partial(p^2 I_{n,1})}{\partial p^2}=\frac{1}{n+1-\epsilon}F\left(1+\epsilon\,,\,2\,;\,n+2-\epsilon\,;\,\frac{p^2}{m^2}\right)\,.$$

Taking the derivative of eq. (75), it follows that

$$\frac{\partial A}{\partial p^2}=\frac{\alpha\,\xi}{2\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{2-\epsilon}\left\{F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)-\left(\frac{1-\epsilon}{3-\epsilon}\right)F\left(1+\epsilon\,,\,2\,;\,4-\epsilon\,;\,\frac{p^2}{m^2}\right)\right\}.$$

Likewise, we can compute $\partial B/\partial p^2$ by taking the derivative of eq. (76). We can simplify the expression for $\partial A/\partial p^2$ by using the recursion relation,

$$(c-b-1)F(a\,,\,b\,;\,c\,;\,z)+bF(a\,,\,b+1\,;\,c\,;\,z)-(c-1)F(a\,,\,b\,;\,c-1\,;\,z)=0\,.$$

The end result is

$$\frac{\partial A}{\partial p^2}=\frac{\alpha\,\xi}{\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{(2-\epsilon)(3-\epsilon)}\,F\left(1+\epsilon\,,\,3\,;\,4-\epsilon\,;\,\frac{p^2}{m^2}\right)\,,\tag{80}$$

$$\frac{\partial B}{\partial p^2}=\frac{\alpha}{2\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{(1-\epsilon)(2-\epsilon)}\left[\tfrac{1}{2}(3+\xi)-\epsilon\right]F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\,.\tag{81}$$

The infrared divergence emerges when $p^2=m^2$. In this limit, we can employ the identity,

$$F(a\,,\,b\,;\,c\,;\,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,,$$

in eqs. (80) and (81) to recover the results of eqs. (50) and (51).

Finally, to demonstrate the fact that the infrared divergence in δZ_2 cancels in the Yennie gauge, we make use of another recursion relation,

$$cF(a\,,\,b\,;\,c\,;\,z)-bzF(a\,,\,b+1\,;\,c+1\,;\,z)-cF(a-1\,,\,b\,;\,c\,;\,z)=0\,,$$

to write

$$(3-\epsilon)\left[F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)-F\left(\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\right]-\frac{2p^2}{m^2}F\left(1+\epsilon\,,\,3\,;\,4-\epsilon\,;\,\frac{p^2}{m^2}\right)=0\,.$$

Hence, it follows that

$$\begin{aligned}\frac{\partial B}{\partial p^2}-\frac{\partial A}{\partial p^2}&=\frac{\alpha}{2\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{2-\epsilon}\left\{\left[\frac{\frac{1}{2}(3+\xi)-\epsilon}{1-\epsilon}-\frac{\xi m^2}{p^2}\right]F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\right.\\&\qquad\left.+\frac{\xi m^2}{p^2}F\left(\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\right\}\\&=\frac{\alpha}{4\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{(1-\epsilon)(2-\epsilon)}\left[3-\xi-2\epsilon(1-\xi)+\xi\left(1-\frac{m^2}{p^2}\right)\right]F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\\&\qquad+\frac{2\xi(1-\epsilon)m^2}{p^2}F\left(\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\Big\}\,.\end{aligned}$$

In the case of $\xi = 3$, we can take the $\epsilon \to 0$ limit when $p^2 = m^2$ without encountering an infrared divergence in δZ_2,

$$\lim_{p^2\to m^2}\left(\frac{\partial B}{\partial p^2}-\frac{\partial A}{\partial p^2}\right)_{\xi=3} = \frac{\alpha}{8\pi m^2}\left\{6+\lim_{\epsilon\to 0}4\epsilon F(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,1)\right\} = \frac{\alpha}{4\pi m^2}\,.$$

That is, in light of eq. (44), we recover eq. (56) in the Yennie gauge,

$$(\delta Z_2^{\rm OS}-\delta Z_2^{\overline{\rm MS}})_{\xi=3} = -\frac{\alpha}{\pi}\left[1-\tfrac{3}{4}\ln\left(\frac{m^2}{\mu^2}\right)\right]\,.$$

Appendix A Integrals arising in one-loop calculations

A.1 The formulae of dimensional regularization

In dimensional regularization, loop integrals are carried out in $d = 4-2\epsilon$ dimensions, which defines the parameter ϵ. In particular,

$$g^{\mu\nu}g_{\mu\nu} = d = 4-2\epsilon\,. \tag{A.1}$$

Ultraviolet and infrared divergences will appear as poles in ϵ.

$$\begin{aligned}
\int\frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+2q\!\cdot\!p-m^2+i\varepsilon)^r} &= i(-1)^r(p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\,\frac{\Gamma(\epsilon+r-2)}{\Gamma(r)} \\
\int\frac{d^dq}{(2\pi)^d}\,\frac{q^\mu}{(q^2+2q\!\cdot\!p-m^2+i\varepsilon)^r} &= -i(-1)^r(p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\,\frac{\Gamma(\epsilon+r-2)}{\Gamma(r)}\,p^\mu \\
\int\frac{d^dq}{(2\pi)^d}\,\frac{q^\mu q^\nu}{(q^2+2q\!\cdot\!p-m^2+i\varepsilon)^r} &= i(-1)^r(p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\,\frac{\Gamma(\epsilon+r-3)}{\Gamma(r)} \\
&\quad\times\left[(\epsilon+r-3)p^\mu p^\nu-\tfrac{1}{2}g^{\mu\nu}(p^2+m^2)\right] \\
\int\frac{d^dq}{(2\pi)^d}\,\frac{q^\mu q^\nu q^\alpha}{(q^2+2q\!\cdot\!p-m^2+i\varepsilon)^r} &= -i(-1)^r(p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\,\frac{\Gamma(\epsilon+r-3)}{\Gamma(r)} \\
&\quad\times\big[(\epsilon+r-3)p^\mu p^\nu p^\alpha \\
&\qquad-\tfrac{1}{2}(g^{\mu\nu}p^\alpha+g^{\mu\alpha}p^\nu+g^{\nu\alpha}p^\mu)(p^2+m^2)\big] \\
\int\frac{d^dq}{(2\pi)^d}\,\frac{q^\mu q^\nu q^\alpha q^\beta}{(q^2+2q\!\cdot\!p-m^2+i\varepsilon)^r} &= i(-1)^r(p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\,\frac{\Gamma(\epsilon+r-4)}{\Gamma(r)} \\
&\quad\times\Big\{(\epsilon+r-3)(\epsilon+r-4)p^\mu p^\nu p^\alpha p^\beta \\
&\qquad-\tfrac{1}{2}(\epsilon+r-4)(g^{\mu\nu}p^\alpha p^\beta+g^{\mu\alpha}p^\nu p^\beta+g^{\mu\beta}p^\nu p^\alpha \\
&\qquad+g^{\nu\alpha}p^\mu p^\beta+g^{\nu\beta}p^\mu p^\alpha+g^{\alpha\beta}p^\mu p^\nu)(p^2+m^2) \\
&\qquad+\tfrac{1}{4}(g^{\mu\nu}g^{\alpha\beta}+g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha})(p^2+m^2)^2\Big\}
\end{aligned}$$

where ε (not to be confused with ϵ) is a positive infinitesimal constant.

In addition, all scaleless integrals are *defined* by the dimensionless regularization procedure to be zero. For example,

$$\int \frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2)^r} \equiv 0\,, \tag{A.2}$$

which corresponds to setting $p^2 = m^2 = 0$ in the first integral above under the assumption that $\epsilon < 2 - r$. However, in dimensional regularization, eq. (A.2) is defined to be valid for *all* r.

We can expand about $\epsilon = 0$ by using

$$\Gamma(-N+\epsilon) = \frac{(-1)^N}{N!}\left[\frac{1}{\epsilon} + \psi(N+1) + \mathcal{O}(\epsilon)\right]\,, \tag{A.3}$$

where N is a non-negative integer, $\psi(x) \equiv \Gamma'(x)/\Gamma(x)$ with $\Gamma'(x) \equiv d\Gamma(x)/dx$,

$$\psi(1) = -\gamma\,, \qquad\qquad \psi(N+1) = -\gamma + \sum_{k=1}^{N}\frac{1}{k}\,, \tag{A.4}$$

and $\gamma = -\Gamma'(1) = 0.5772\cdots$ is the Euler-Mascheroni constant.

Finally, we record some of the Feynman parameter formulae:

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx\,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\big[xA+(1-x)B\big]^{\alpha+\beta}}\,, \tag{A.5}$$

$$\frac{1}{A^\alpha B^\beta C^\delta} = \frac{\Gamma(\alpha+\beta+\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta)}\int_0^1 x\,dx\int_0^1 dy\,\frac{x^{\alpha+\beta-2}\,y^{\alpha-1}(1-x)^{\delta-1}(1-y)^{\beta-1}}{\big[xyA+x(1-y)B+(1-x)C\big]^{\alpha+\beta+\delta}}\,, \tag{A.6}$$

and more generally,

$$\frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}\cdots A_N^{\alpha_N}} = \frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_N)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_N)}\int_0^1 dx_1\cdots\int_0^1 dx_N\,\delta\left(\sum_{j=1}^N x_j - 1\right)$$
$$\times\frac{x_1^{\alpha_1-1}\,x_2^{\alpha_2-1}\cdots x_N^{\alpha_N-1}}{(x_1A_1+x_2A_2+\cdots+x_NA_N)^{\alpha_1+\alpha_2+\cdots+\alpha_N}}\,. \tag{A.7}$$

A.2 The Passarino–Veltman loop functions

We collect here the relevant integrals that arise in one-loop computations of one-point, two-point and three-point Green functions based on work that first appeared in Ref. [**?**, **?**]. However, in contrast to the original presentation, we employ the metric convention $g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$, and we have chosen a different overall normalization constant in defining the loop functions. Here, we follow the conventions that appear in Ref. [**?**].

The Passarino–Veltman loop functions that arise when evaluating the one loop contributions to one-point, two-point and three-point Green functions (or the corresponding amplitudes of a physical process) are defined as follows,[1]

$$A_0(m^2) = -16\pi^2 i\mu^{2\epsilon}\int \frac{d^d q}{(2\pi)^d}\frac{1}{q^2-m^2+i\varepsilon}, \tag{A.8}$$

$$B_0; B^\mu; B^{\mu\nu}(p^2; m_a^2, m_b^2) = -16\pi^2 i\mu^{2\epsilon}\int \frac{d^d q}{(2\pi)^d}\frac{1; q^\mu; q^\mu q^\nu}{D_B}, \tag{A.9}$$

$$C_0; C^\mu; C^{\mu\nu}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = -16\pi^2 i\mu^{2\epsilon}\int \frac{d^d q}{(2\pi)^d}\frac{1; q^\mu; q^\mu q^\nu}{D_C}, \tag{A.10}$$

where the integrals are evaluated in $d = 4 - 2\epsilon$ dimensions,

$$D_B \equiv (q^2 - m_a^2 + i\varepsilon)[(q+p)^2 - m_b^2 + i\varepsilon]\,, \tag{A.11}$$

$$D_C \equiv (q^2 - m_a^2 + i\varepsilon)[(q+p_1)^2 - m_b^2 + i\varepsilon][(q+p_1+p_2)^2 - m_c^2 + i\varepsilon]\,, \tag{A.12}$$

and $p = -(p_1 + p_2)$. In Eqs. (A.8)-(A.10) all external momenta are flowing into the diagrammatic representation of the Green function. We have included the $\mu^{2\epsilon}$ factor for convenience as it will ensure that the arguments of all logarithms that arise in the evaluation of the above integrals are dimensionless.

The arguments of the Passarino–Veltman loop functions employed in eqs. (A.9) and (A.10) have been chosen with the understanding that Lorentz covariance can be used to decompose the loop functions in terms of Lorentz scalar functions of the same arguments,

$$B^\mu = B_1 p^\mu\,, \tag{A.13}$$

$$B^{\mu\nu} = B_{21} p^\mu p^\nu + B_{22} g^{\mu\nu}\,, \tag{A.14}$$

$$C^\mu = C_{11} p_1^\mu + C_{12} p_2^\mu\,, \tag{A.15}$$

$$C^{\mu\nu} = C_{21} p_1^\mu p_1^\nu + C_{22} p_2^\mu p_2^\nu + C_{23}(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) + C_{24} g^{\mu\nu}\,. \tag{A.16}$$

The derivatives of B-type integrals are also of interest and will be analyzed below.[2]

Among the integrals listed above, A_0, B_0, B_1, B_{21}, B_{22} and C_{24} are divergent as $\epsilon \to 0$. The integrals C_0 and C_{ij} for $ij \neq 24$ are ultraviolet convergent and can be evaluated by setting $\epsilon = 0$ (assuming that no infrared divergences are present). The divergent parts of A_0, B_0, B_1 and C_{24} are easily obtained,

$$A_0(m^2)|_{\rm div} = \frac{m^2}{\epsilon} \tag{A.17}$$

$$B_0(p^2; m_a^2, m_b^2)|_{\rm div} = \frac{1}{\epsilon} \tag{A.18}$$

$$B_1(p^2; m_a^2, m_b^2)|_{\rm div.} = -\frac{1}{2\epsilon} \tag{A.19}$$

$$C_{24}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2)|_{\rm div.} = \frac{1}{4\epsilon}\,. \tag{A.20}$$

[1]All squared masses, m^2, m_a^2, m_b^2 and m_c^2 are nonnegative real parameters. The four momenta p_1, p_2 and p correspond to either on-shell or off-shell particles depending on the application.

[2]One can also consider B-type and C-type tensor integrals with more than two Lorentz indices, the derivatives of C-type integrals, and the D-type and E-type Passarino-Veltman loop functions that arise when evaluating the respective one-loop contributions to four-point and five-point Green functions. These loop integrals will not be treated in this Appendix; for further details, the reader may consult Refs. [**?**].

It is convenient to introduce the quantity,

$$\Delta \equiv (4\pi)^\epsilon\,\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \ln(4\pi) + \mathcal{O}(\epsilon)\,, \tag{A.21}$$

where γ is Euler's constant. Then, we can evaluate A_0, B_0 and B_1 explicitly in $d = 4 - 2\epsilon$ dimensions. All terms of $\mathcal{O}(\epsilon)$ will be dropped in the expressions below. The following results are then obtained,

$$A_0(m^2) = m^2\left[\Delta + 1 - \ln\left(\frac{m^2}{\mu^2}\right)\right]\,, \tag{A.22}$$

$$B_0(p^2;m_a^2,m_b^2) = \Delta - \int_0^1 \ln\left(\frac{p^2x^2 - (p^2+m_a^2-m_b^2)x + m_a^2 - i\varepsilon}{\mu^2}\right) dx\,, \tag{A.23}$$

$$B_1(p^2;m_a^2,m_b^2) = -\tfrac{1}{2}\Delta + \int_0^1 \ln\left(\frac{p^2x^2 - (p^2+m_a^2-m_b^2)x + m_a^2 - i\varepsilon}{\mu^2}\right) x\,dx\,, \tag{A.24}$$

$$B_{21}(p^2;m_a^2,m_b^2) = \tfrac{1}{3}\Delta - \int_0^1 \ln\left(\frac{p^2x^2 - (p^2+m_a^2-m_b^2)x + m_a^2 - i\varepsilon}{\mu^2}\right) x^2\,dx\,, \tag{A.25}$$

$$\begin{aligned} B_{22}(p^2;m_a^2,m_b^2) = \tfrac{1}{4}(\Delta+1)\bigl(m_a^2+m_b^2-\tfrac{1}{3}p^2\bigr) - \tfrac{1}{2}\int_0^1 \bigl[p^2x^2 - (p^2+m_a^2-m_b^2)x + m_a^2\bigr] \\ \times \ln\left(\frac{p^2x^2 - (p^2+m_a^2-m_b^2)x + m_a^2 - i\varepsilon}{\mu^2}\right) dx\,. \end{aligned} \tag{A.26}$$

It is possible to evaluate B_1 in terms of A_0 and B_0 by noting that ,

$$p^2 B_1(p^2;m_a^2,m_b^2) = p_\mu B^\mu(p^2;m_a^2,m_b^2) = -16\pi^2 i\mu^{2\epsilon}\int \frac{d^dq}{(2\pi)^d}\,\frac{p\!\cdot\! q}{D_B}\,, \tag{A.27}$$

where D_B is given in eq. (A.11). To simplify this result, we shall employ the method of partial fractions by making use of following algebraic identity,

$$p\!\cdot\! q = \tfrac{1}{2}\bigl[(q+p)^2 - q^2 - p^2\bigr] = \tfrac{1}{2}\bigl[(q+p)^2 - m_b^2 - (q^2 - m_a^2) - p^2 + m_b^2 - m_a^2\bigr]\,. \tag{A.28}$$

Plugging this result into eq. (A.27) yields,

$$\begin{aligned} p^2 B_1(p^2;m_a^2,m_b^2) = 8\pi^2 i\mu^{2\epsilon}\biggl\{\int \frac{d^dq}{(2\pi)^d}\left(\frac{1}{(q+p)^2 - m_b^2 + i\varepsilon} - \frac{1}{q^2 - m_a^2 + i\varepsilon}\right) \\ +(p^2+m_a^2-m_b^2)\mu^{2\epsilon}\int \frac{d^dq}{(2\pi)^d}\,\frac{1}{D_B}\biggr\}\,. \end{aligned} \tag{A.29}$$

The end result is,

$$p^2 B_1(p^2;m_a^2,m_b^2) = \tfrac{1}{2}\bigl[A_0(m_a^2) - A_0(m_b^2) - (p^2+m_a^2-m_b^2)B_0(p^2;m_a^2,m_b^2)\bigr]\,. \tag{A.30}$$

In particular,

$$B_1(p^2;m^2,m^2) = -\tfrac{1}{2}B_0(p^2;m^2,m^2)\,. \tag{A.31}$$

Similarly, B_{21} and B_{22} can be expressed in terms of A_0, B_0 and B_1. Starting from eq. (A.14), it follows that

$$p^\mu B_{\mu\nu} = p_\nu(p^2 B_{21} + B_{22})\,, \qquad\qquad g^{\mu\nu} B_{\mu\nu} = p^2 B_{21} + dB_{22}\,, \tag{A.32}$$

where the arguments of the B loop functions are $(p^2; m_a^2, m_b^2)$. Following the steps used in the derivation of eq. (A.30), we obtain two equations,

$$p^2 B_{21} + B_{22} = \tfrac{1}{2} A_0(m_b^2) - \tfrac{1}{2}(p^2 + m_a^2 - m_b^2) B_1\,, \tag{A.33}$$

$$p^2 B_{21} + dB_{22} = A_0(m_b^2) + m_a^2 B_0\,. \tag{A.34}$$

Solving for B_{21} and B_{22} yields,

$$(d-1)p^2 B_{21} = \left(\tfrac{1}{2}d - 1\right) A_0(m_b^2) - \tfrac{1}{2}d(p^2 + m_a^2 - m_b^2) B_1 - m_a^2 B_0\,, \tag{A.35}$$

$$(d-1) B_{22} = \tfrac{1}{2} A_0(m_b^2) + m_a^2 B_0 + \tfrac{1}{2}(p^2 + m_a^2 - m_b^2) B_1\,. \tag{A.36}$$

After expanding about $\epsilon = 0$ and dropping terms of $\mathcal{O}(\epsilon)$, it follows that

$$dB_{21} = 4B_{21} - \tfrac{2}{3}\,, \qquad\qquad dB_{22} = 4B_{22} - \tfrac{1}{2}\left(m_a^2 + m_b^2 - \tfrac{1}{3}p^2\right), \tag{A.37}$$

$$dB_1 = 4B_1 + 1\,, \qquad\qquad \tfrac{1}{2} dA_0(m_b^2) = 2A_0(m_b^2) - m_b^2\,. \tag{A.38}$$

We end up with

$$\begin{aligned} p^2 B_{21}(p^2; m_a^2, m_b^2) = \frac{1}{3}\bigg\{ & A_0(m_b^2) - m_a^2 B_0(p^2; m_a^2, m_b^2) - \tfrac{1}{2}(m_a^2 + m_b^2) + \frac{p^2}{6} \\ & - 2(p^2 + m_a^2 - m_b^2) B_1(p^2; m_a^2, m_b^2)\bigg\}, \end{aligned} \tag{A.39}$$

$$\begin{aligned} B_{22}(p^2; m_a^2, m_b^2) = \frac{1}{6}\bigg\{ & A_0(m_b^2) + 2m_a^2 B_0(p^2; m_a^2, m_b^2) + m_a^2 + m_b^2 - \frac{p^2}{3} \\ & + (p^2 + m_a^2 - m_b^2) B_1(p^2; m_a^2, m_b^2)\bigg\}. \end{aligned} \tag{A.40}$$

The symmetry properties of B_0, B_1, B_{21} and B_{22} under an interchange of $m_a^2 \leftrightarrow m_b^2$ are noteworthy,

$$B_0(p^2; m_a^2, m_b^2) = B_0(p^2; m_b^2, m_a^2)\,, \tag{A.41}$$

$$B_1(p^2; m_a^2, m_b^2) = -B_1(p^2; m_b^2, m_a^2) - B_0(p^2; m_b^2, m_a^2)\,. \tag{A.42}$$

$$B_{21}(p^2; m_a^2, m_b^2) = B_{21}(p^2; m_b^2, m_b^2) + 2B_{11}(p^2; m_b^2, m_b^2) + B_0(p^2; m_b^2, m_b^2)\,, \tag{A.43}$$

$$B_{22}(p^2; m_a^2, m_b^2) = B_{22}(p^2; m_b^2, m_a^2)\,. \tag{A.44}$$

One can now perform the integration in eq. (A.23). The end result is,

$$B_0(p^2; m_a^2, m_b^2) = \Delta - F(p^2; m_a^2, m_b^2)\,, \tag{A.45}$$

where the function F is explicitly evaluated below in five distinct cases [?]. The expressions make use of the well-known kinematical triangle function [?],

$$\lambda(a^2, b^2, c^2) \equiv a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = \left[a^2 - (b+c)^2\right]\left[a^2 - (b-c)^2\right]. \tag{A.46}$$

Case 1: $p^2 > (m_a + m_b)^2$

$$F(p^2; m_a^2, m_b^2) = \ln\left(\frac{m_b^2}{\mu^2}\right) - 2 - \left(\frac{p^2 + m_a^2 - m_b^2}{2p^2}\right)\ln\left(\frac{m_b^2}{m_a^2}\right) \tag{A.47}$$
$$+\frac{\lambda^{1/2}(p^2, m_a^2, m_b^2)}{p^2}\left[\ln\left(\frac{\left[p^2 - (m_a - m_b)^2\right]^{1/2} + \left[p^2 - (m_a + m_b)^2\right]^{1/2}}{\left[p^2 - (m_a - m_b)^2\right]^{1/2} - \left[p^2 - (m_a + m_b)^2\right]^{1/2}}\right) - i\pi\right].$$

Case 2: $p^2 < (m_a - m_b)^2$ and $p^2 \neq 0$

$$F(p^2; m_a^2, m_b^2) = \ln\left(\frac{m_b^2}{\mu^2}\right) - 2 - \left(\frac{p^2 + m_a^2 - m_b^2}{2p^2}\right)\ln\left(\frac{m_b^2}{m_a^2}\right) \tag{A.48}$$
$$+\frac{\lambda^{1/2}(p^2, m_a^2, m_b^2)}{p^2}\ln\left(\frac{\left[(m_a + m_b)^2 - p^2\right]^{1/2} + \left[(m_a - m_b)^2 - p^2\right]^{1/2}}{\left[(m_a + m_b)^2 - p^2\right]^{1/2} - \left[(m_a - m_b)^2 - p^2\right]^{1/2}}\right).$$

Case 3: $(m_a - m_b)^2 < p^2 < (m_a + m_b)^2$

$$F(p^2; m_a^2, m_b^2) = \ln\left(\frac{m_b^2}{\mu^2}\right) - \left(\frac{p^2 + m_a^2 - m_b^2}{2p^2}\right)\ln\left(\frac{m_b^2}{m_a^2}\right) - 2$$
$$+\frac{2\left[-\lambda(p^2, m_a^2, m_b^2)\right]^{1/2}}{p^2}\arctan\left(\frac{\sqrt{p^2 - (m_a - m_b)^2}}{\sqrt{(m_a + m_b)^2 - p^2}}\right), \tag{A.49}$$

where the principal value of the real arctangent function satisfies $|\arctan x| \leq \frac{1}{2}\pi$.

Case 4: $p^2 = 0$ and $m_a \neq m_b$

$$F(0; m_a^2, m_b^2) = \frac{1}{m_a^2 - m_b^2}\left[m_a^2 \ln\left(\frac{m_a^2}{\mu^2}\right) - m_b^2 \ln\left(\frac{m_b^2}{\mu^2}\right)\right] - 1\,. \tag{A.50}$$

Case 5: $p^2 = 0$ and $m \equiv m_a = m_b$

$$F(0; m^2, m^2) = \ln\left(\frac{m^2}{\mu^2}\right)\,. \tag{A.51}$$

Sometimes a loop integral arises in which one or more of the propagator denominators are raised to a power. For an A-type loop integral, we can simply use the formulae provided in Appendix A.1. As an example,

For a B-type loop integral, consider

$$\begin{aligned}\frac{\partial B_0}{\partial m_a^2}(p^2; m_a^2, m_b^2) &= -16\pi^2 i\mu^{2\epsilon}\int\frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2 - m_a^2 + i\varepsilon)^2[(q+p)^2 - m_b^2 + i\varepsilon]}\\ &= -\int_0^1 \frac{(1-x)dx}{p^2x^2 - (p^2 + m_a^2 - m_b^2)x + m_a^2 - i\varepsilon}\,.\end{aligned} \tag{A.52}$$

One can express this integral in terms of A_0 and B_0 as follows,

$$\frac{\partial B_0}{\partial m_a^2}(p^2;m_a^2,m_b^2) = \frac{1}{\lambda(p^2,m_a^2,m_b^2)}\biggl\{(-p^2+m_a^2-m_b^2)B_0(p^2;m_a^2,m_b^2) + \left(\frac{p^2-m_a^2-m_b^2}{m_a^2}\right)A_0(m_a^2)+2A_0(m_b^2)+p^2-m_a^2+m_b^2\biggr\}, \tag{A.53}$$

under the assumption that $\lambda(p^2,m_a^2,m_b^2)\neq 0$ [cf. eq. (A.46)].

Taking the derivative of eq. (A.30) with respect to m_a^2 yields,

$$p^2\frac{\partial B_1}{\partial m_a^2}(p^2;m_a^2,m_b^2) = \frac{1}{2}\left[\frac{A_0(m_a^2)}{m_a^2}-1-B_0(p^2;m_a^2,m_b^2)-(p^2-m_a^2+m_b^2)\frac{\partial B_0}{\partial m_a^2}(p^2;m_a^2,m_b^2)\right], \tag{A.54}$$

after employing

$$\frac{\partial A_0(m_a^2)}{\partial m_a^2} = \Delta - \ln\left(\frac{m^2}{\mu^2}\right) = \frac{A_0(m_a^2)}{m_a^2}-1\,, \tag{A.55}$$

in light of eqs. (A.8) and (A.22) after dropping terms of $\mathcal{O}(\epsilon)$. Likewise, taking the derivative of eqs. (A.35) and (A.36) yields,

$$-p^2(d-1)\frac{\partial B_{21}}{\partial m_a^2}(p^2;m_a^2,m_b^2) = B_0(p^2;m_a^2,m_b^2)+\tfrac{1}{2}dB_1(p^2;m_a^2,m_b^2)+m_a^2\,\frac{\partial B_0}{\partial m_a^2}(p^2;m_a^2,m_b^2) + \tfrac{1}{2}d(p^2+m_a^2-m_b^2)\frac{\partial B_1}{\partial m_a^2}(p^2;m_a^2,m_b^2)\,, \tag{A.56}$$

$$p^2(d-1)\frac{\partial B_{22}}{\partial m_a^2}(p^2;m_a^2,m_b^2) = B_0(p^2;m_a^2,m_b^2)+\tfrac{1}{2}B_1(p^2;m_a^2,m_b^2)+m_a^2\,\frac{\partial B_0}{\partial m_a^2}(p^2;m_a^2,m_b^2) + \tfrac{1}{2}(p^2+m_a^2-m_b^2)\frac{\partial B_1}{\partial m_a^2}(p^2;m_a^2,m_b^2)\,. \tag{A.57}$$

If only ultraviolet divergences are present, then we can employ $\frac{1}{2}dB_1=2B_1+\frac{1}{2}$ and

$$(d-1)\frac{\partial B_{22}}{\partial m_a^2}(p^2;m_a^2,m_b^2) = 3\frac{\partial B_{22}}{\partial m_a^2}(p^2;m_a^2,m_b^2)-\frac{1}{2}\,, \tag{A.58}$$

while setting $d=4$ everywhere else to obtain,

$$-p^2\frac{\partial B_{21}}{\partial m_a^2}(p^2;m_a^2,m_b^2) = \frac{1}{3}\biggl[B_0(p^2;m_a^2,m_b^2)+2B_1(p^2;m_a^2,m_b^2)+\frac{1}{2}+m_a^2\,\frac{\partial B_0}{\partial m_a^2}(p^2;m_a^2,m_b^2) + 2(p^2+m_a^2-m_b^2)\frac{\partial B_1}{\partial m_a^2}(p^2;m_a^2,m_b^2)\biggr], \tag{A.59}$$

$$p^2\frac{\partial B_{22}}{\partial m_a^2}(p^2;m_a^2,m_b^2) = \frac{1}{3}\biggl[B_0(p^2;m_a^2,m_b^2)+\tfrac{1}{2}B_1(p^2;m_a^2,m_b^2)+\frac{1}{2}+m_a^2\,\frac{\partial B_0}{\partial m_a^2}(p^2;m_a^2,m_b^2) + \tfrac{1}{2}(p^2+m_a^2-m_b^2)\frac{\partial B_1}{\partial m_a^2}(p^2;m_a^2,m_b^2)\biggr]. \tag{A.60}$$

In computing wave function renormalization, one encounters derivatives of B-type loop integrals with respect to p^2. For example, taking the derivative with respect to p^2 of eq. (A.23) yields,

$$B_0'(p^2;m_a^2,m_b^2) \equiv \frac{\partial}{\partial p^2} B_0(p^2;m_a^2,m_b^2) = \int_0^1 \frac{x(1-x)dx}{p^2x^2-(p^2+m_a^2-m_b^2)x+m_a^2-i\varepsilon}\,. \tag{A.61}$$

Note that,

$$B_0'(p^2;m_a^2,m_b^2) = B_0'(p^2;m_b^2,m_a^2)\,. \tag{A.62}$$

One can obtain the following expression for B_0' in terms of $B_0 \equiv B_0(p^2;m_a^2,m_b^2)$ and A_0,

$$\begin{aligned} B_0'(p^2;m_a^2,m_b^2) = \frac{1}{p^2\lambda(p^2,m_a^2,m_b^2)}\Big\{ &\big[p^2(m_a^2+m_b^2)-(m_b^2-m_a^2)^2\big]B_0 \\ &-A_0(m_a^2)[p^2-m_a^2+m_b^2]-A_0(m_b^2)[p^2+m_a^2-m_b^2]+p^2(m_a^2+m_b^2-p^2)\Big\}\,, \end{aligned} \tag{A.63}$$

under the assumption that $p^2 \neq 0$ and $\lambda(p^2,m_a^2,m_b^2) \neq 0$ [cf. eq. (A.46)].

Taking the derivatives with respect to p^2 of eqs. (A.30), (A.39) and (A.40) yields,

$$B_1'(p^2;m_a^2,m_b^2) = -\frac{1}{p^2}\big[B_1+\tfrac{1}{2}B_0+\tfrac{1}{2}(p^2+m_a^2-m_b^2)B_0'\big]\,, \tag{A.64}$$

$$B_{21}'(p^2;m_a^2,m_b^2) = -\frac{1}{3p^2}\left[3B_{21}+2B_1-\tfrac{1}{6}+m_a^2B_0'+2(p^2+m_a^2-m_b^2)B_1'\right]\,, \tag{A.65}$$

$$B_{22}'(p^2;m_a^2,m_b^2) = \tfrac{1}{6}\big[B_1-\tfrac{1}{3}+2m_a^2B_0'+(p^2+m_a^2-m_b^2)B_1'\big]\,, \tag{A.66}$$

where the suppressed arguments of the loop functions above are $(p^2;m_a^2,m_b^2)$.

For completeness, we list the following limiting cases,

$$\begin{aligned}
B_0(0;m_a^2,m_b^2) &= \frac{A_0(m_a^2)-A_0(m_b^2)}{m_a^2-m_b^2}\,,\\
B_0(0;m^2,m^2) &= \frac{A_0(m^2)}{m^2}-1\,,\\
B_1(0,m_a^2,m_b^2) &= -\frac{1}{4(m_a^2-m_b^2)^2}\big[m_a^4-m_b^4+2m_a^2A_0(m_a^2)-2(2m_a^2-m_b^2)A_0(m_b^2)\big]\,,\\
B_1(0;m^2,m^2) &= -\frac{1}{2}\left(\frac{A_0(m^2)}{m^2}-1\right)\,,\\
\frac{\partial B_0}{\partial m_a^2}(0,m_a^2,m_b^2) &= -\frac{1}{m_a^2-m_b^2}-\frac{m_b^2}{(m_a^2-m_b^2)^2}\left(\frac{A_0(m_a^2)}{m_a^2}-\frac{A_0(m_b^2)}{m_b^2}\right),\\
\frac{\partial B_0}{\partial m_a^2}(0,m^2,m^2) &= -\frac{1}{2m^2}\,,\\
\frac{\partial B_0}{\partial m_a^2}\big((m_a-m_b)^2;m_a^2,m_b^2\big) &= \frac{1}{m_a(m_a-m_b)}+\frac{1}{2(m_a-m_b)^2}\left(\frac{A_0(m_a^2)}{m_a^2}-\frac{A_0(m_b^2)}{m_a^2}\right),\\
B_0'(0;m_a^2,m_b^2) &= \frac{1}{(m_a^2-m_b^2)^3}\left[\tfrac{1}{2}(m_a^4-m_b^4)+m_b^2A_0(m_a^2)-m_a^2A_0(m_b^2)\right]\,,\\
B_0'(0;m^2,m^2) &= \frac{1}{6m^2}\,,\\
B_0'\big((m_a-m_b)^2;m_a^2,m_b^2\big) &= -\frac{2}{(m_a-m_b)^2}-\frac{m_a+m_b}{2(m_a-m_b)^3}\left(\frac{A_0(m_a^2)}{m_a^2}-\frac{A_0(m_b^2)}{m_b^2}\right).
\end{aligned} \tag{A.67}$$

Note that if $\lambda(p^2, m_a^2, m_b^2) = 0$, then it follows that $p^2 = (m_a + m_b)^2$ or $p^2 = (m_a - m_b)^2$. Evaluating $B_0'(p^2; m_a^2, m_b^2)$ and $\partial B_0(p^2; m_a^2, m_b^2)/\partial m_a^2$ at $p^2 = (m_a - m_b)^2$ requires some care due to the presence of an infrared divergence.

First, consider the following expression for $B_0(p^2; m_a^2, m_b^2)$,

$$B_0(p^2; m_a^2, m_b^2) = (4\pi\mu^2)^\epsilon\, \Gamma(\epsilon) \int_0^1 \left[p^2x^2 - (p^2 + m_a^2 - m_b^2)x + m_a^2 - i\varepsilon\right]^{-\epsilon} dx\,. \tag{A.68}$$

Infrared divergences, if present, will reveal themselves when performing the integration over x. iI no infrared divergences are present, then one may perform an expansion in ϵ to obtain the integral representation given in eq. (A.23). Here, we shall postpone the expansion in ϵ and work to all orders in ϵ until the penultimate step of the computation. Differentiating eq. (A.68) with respect to p^2 yields,

$$B_0'(p^2; m_a^2, m_b^2) = (4\pi\mu^2)^\epsilon\, \Gamma(1+\epsilon) \int_0^1 \left[p^2x^2 - (p^2 + m_a^2 - m_b^2)x + m_a^2 - i\varepsilon\right]^{-1-\epsilon} x(1-x)dx\,, \tag{A.69}$$

after using $\epsilon\Gamma(\epsilon) = \Gamma(1+\epsilon)$. Plugging in $p^2 = (m_a + m_b)^2$ yields,

$$B_0'\big((m_a+m_b)^2; m_a^2, m_b^2\big) = (4\pi\mu^2)^\epsilon\, \Gamma(1+\epsilon)\, \mathcal{I}(m_a, m_b)\,. \tag{A.70}$$

where

$$\mathcal{I}(m_a, m_b) \equiv \int_0^1 \Big(\big[m_a - (m_a+m_b)x\big]^2\Big)^{-1-\epsilon} x(1-x)\, dx\,. \tag{A.71}$$

The integral $\mathcal{I}(m_a, m_b)$ is well defined if $\operatorname{Re}\epsilon < -\frac{1}{2}$. After integration, we shall analytically continue the result in the complex ϵ-plane to the region near $\epsilon = 0$. Note that the fact that the analytic continuation is performed starting from the negative region of $\operatorname{Re}\epsilon$ is the hallmark of the infrared divergence, which will be exposed at $\epsilon = 0$. To evaluate $\mathcal{I}(m_a, m_b)$, we shall break up the integration range into two intervals, $0 < x < m_a/(m_a+m_b)$ and $m_a/(m_a+m_b) < x < 1$. In the first interval we redefine the integration variable by $y = x(m_a + m_b)/m_a$, and in the second interval we redefine the integration variable by $y = (1-x)(m_a + m_b)/m_b$. It then follows that

$$\begin{aligned}\mathcal{I}(m_a, m_b) = \frac{1}{(m_a+m_b)^2} &\int_0^1 y(1-y)^{-2-2\epsilon} \\ &\times \left\{(m_a^2)^{-\epsilon}\left[1 - \left(\frac{m_a}{m_a+m_b}\right)y\right] + (m_b^2)^{-\epsilon}\left[1 - \left(\frac{m_b}{m_a+m_b}\right)y\right]\right\} dy,\end{aligned} \tag{A.72}$$

where $(1-y)^{-2-2\epsilon} \equiv \left[(1-y)^2\right]^{-1-\epsilon}$. Note that $\mathcal{I}(m_a, m_b)$ can be expressed as a sum of Beta functions. Is is straightforward to obtain,

$$\begin{aligned}\mathcal{I}(m_a, m_b) = \frac{1}{(m_a+m_b)^2}\Bigg\{&(m_a^2)^{-\epsilon}\left[B(2, -1-2\epsilon) - \left(\frac{m_a}{m_a+m_b}\right)B(3, -1-2\epsilon)\right] \\ &+ (m_b^2)^{-\epsilon}\left[B(2, -1-2\epsilon) - \left(\frac{m_b}{m_a+m_b}\right)B(3, -1-2\epsilon)\right]\Bigg\}.\end{aligned} \tag{A.73}$$

This result can be simplified by using

$$B(2,-1-2\epsilon)=\frac{\Gamma(2)\Gamma(-1-2\epsilon)}{\Gamma(1-2\epsilon)}=\frac{1}{2\epsilon(1+2\epsilon)}\,,\qquad\qquad B(3,-1-2\epsilon)=\frac{1}{\epsilon(1+2\epsilon)(1-2\epsilon)}\,. \tag{A.74}$$

Hence, we end up with

$$B_0'\big((m_a+m_b)^2;m_a^2,m_b^2\big)=\frac{(4\pi)^\epsilon\Gamma(\epsilon)}{2(1-4\epsilon^2)(m_a+m_b)^2}\times\left\{\left(\frac{m_a^2}{\mu^2}\right)^{-\epsilon}\left[\frac{m_b-m_a}{m_a+m_b}-2\epsilon\right]+\left(\frac{m_b^2}{\mu^2}\right)^{-\epsilon}\left[\frac{m_a-m_b}{m_a+m_b}-2\epsilon\right]\right\}. \tag{A.75}$$

As expected, the above expression is symmetric under the interchange, $m_a \leftrightarrow m_b$. Moreover, as anticipated, the divergence as $\epsilon \to 0$ is a consequence of the infrared divergence that was generated in the integration over the Feynman parameter x.

As a check of the above result, consider the special case of $m_b = 0$. In this case, eq. (A.70) yields,

$$B_0'\big(m_a^2;m_a^2,0\big)=(4\pi)^\epsilon\,\Gamma(1+\epsilon)\left(\frac{m_a^2}{\mu^2}\right)^{-\epsilon}\int_0^1 x(1-x)^{-1-2\epsilon}\,dx\,. \tag{A.76}$$

The integral is easily evaluated,

$$\int_0^1 x(1-x)^{-1-2\epsilon}\,dx=B(2,-2\epsilon)=\frac{\Gamma(-2\epsilon)}{\Gamma(2-2\epsilon)}=-\,\frac{1}{2\epsilon(1-2\epsilon)}\,. \tag{A.77}$$

Hence, in light of eq. (A.62),

$$B_0'(m^2;m^2,0)=B_0'(m^2;0,m^2)=-\,\frac{1}{2m^2(1-2\epsilon)}(4\pi)^\epsilon\,\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\,, \tag{A.78}$$

in agreement with the $m_b = 0$ limit of eq. (A.75) [where $(m_b^2)^{-\epsilon} = 0$ since $\epsilon < 0$]. Moreover, we have reproduced the result of eq. (65).

Second, consider the following expression for $\partial B_0(p^2; m_a^2, m_b^2)/\partial m_a^2$,

$$\frac{\partial B_0}{\partial m_a^2}(p^2;m_a^2,m_b^2)=-(4\pi\mu^2)^\epsilon\,\Gamma(1+\epsilon)\int_0^1\big[p^2x^2-(p^2+m_a^2-m_b^2)x+m_a^2-i\varepsilon\big]^{-1-\epsilon}(1-x)dx\,. \tag{A.79}$$

As previously noted, if no infrared divergences are present, then one may perform an expansion in ϵ to obtain the integral representation given in eq. (A.52). Here, we shall postpone the expansion in ϵ and work to all orders in ϵ until the penultimate step of the computation. Plugging in $p^2 = (m_a + m_b)^2$ yields,

$$\frac{\partial B_0}{\partial m_a^2}\big((m_a+m_b)^2;m_a^2,m_b^2\big)=-(4\pi\mu^2)^\epsilon\,\Gamma(1+\epsilon)\mathcal{J}(m_a,m_b)\,. \tag{A.80}$$

where

$$\mathcal{J}(m_a,m_b)\equiv\int_0^1\Big(\big[m_a-(m_a+m_b)x\big]^2\Big)^{-1-\epsilon}(1-x)\,dx\,. \tag{A.81}$$

The integral $\mathcal{J}(m_a, m_b)$ is well defined if $\operatorname{Re}\epsilon < -\frac{1}{2}$. The computation is nearly identical to the previous one.

$$\begin{aligned}\mathcal{J}(m_a, m_b) &= \frac{m_a^{-1-2\epsilon}}{m_a + m_b} B(1, -1-2\epsilon) + \frac{(m_b^2)^{-\epsilon} - (m_a^2)^{-\epsilon}}{(m_a + m_b)^2} B(2, -1-2\epsilon) \\ &= -\frac{1}{2\epsilon(m_a + m_b)^2} \left\{ (m_a + m_b) m_a^{-1-2\epsilon} + \frac{(m_a^2)^{-\epsilon} - (m_b^2)^{-\epsilon}}{1+2\epsilon} \right\}. \end{aligned} \qquad \text{(A.82)}$$

Hence, we end up with,

$$\frac{\partial B_0}{\partial m_a^2}\big((m_a + m_b)^2; m_a^2, m_b^2\big) = \frac{(4\pi)^\epsilon \Gamma(\epsilon)}{2(m_a + m_b)^2} \left\{ \left(\frac{m_a^2}{\mu^2}\right)^{-\epsilon} \left[1 + \frac{m_b}{m_a}\right] + \frac{1}{1+2\epsilon} \left[\left(\frac{m_a^2}{\mu^2}\right)^{-\epsilon} - \left(\frac{m_b^2}{\mu^2}\right)^{-\epsilon} \right] \right\}. \qquad \text{(A.83)}$$

A special case of interest corresponds to $m_a = 0$. Note that the above expression was derived under the assumption that $\operatorname{Re}\epsilon < -\frac{1}{2}$, which implies that $m_a^{-1-2\epsilon} = 0$. Thus, we obtain,

$$\frac{\partial B_0}{\partial m_a^2}\big((m_a + m_b)^2; m_a^2, m_b^2\big)\bigg|_{m_a=0} = -\frac{1}{2m_b^2(1+2\epsilon)} (4\pi)^\epsilon\, \Gamma(\epsilon) \left(\frac{m_b^2}{\mu^2}\right)^{-\epsilon}. \qquad \text{(A.84)}$$

This can be checked by setting $m_a = 0$ in eqs. (A.80) and (A.81),

$$\frac{\partial B_0}{\partial m_a^2}\big((m_a + m_b)^2; m_a^2, m_b^2\big)\bigg|_{m_a=0} = -\frac{(4\pi)^\epsilon\, \Gamma(1+\epsilon)}{m_b^2} \left(\frac{m_b^2}{\mu^2}\right)^{-\epsilon} \int_0^1 x^{2-2\epsilon}(1-x)dx\,, \qquad \text{(A.85)}$$

which reproduces the result obtained in eq. (A.84).

It is noteworthy that $\partial B_0(p^2; m_a^2, m_b^2)/\partial m_a^2$ at $p^2 = (m_a - m_b)^2$ is a special type of C-type loop integral. In particular, eqs. (A.10) and (A.52) yield,

$$\frac{\partial B_0}{\partial m_a^2}(p^2; m_a^2, m_b^2) = C_0(0, p^2, p^2; m_a^2, m_a^2, m_b^2)\,. \qquad \text{(A.86)}$$

Next, we present integral expressions for C_0 and the C_{ij}.

$$C_0(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = -\int_0^1 dx \int_0^x \frac{dy}{D - i\varepsilon}\,, \qquad \text{(A.87)}$$

$$C_{11}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = \int_0^1 x\, dx \int_0^x \frac{dy}{D - i\varepsilon}\,, \qquad \text{(A.88)}$$

$$C_{12}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = \int_0^1 dx \int_0^x \frac{(x-y)dy}{D - i\varepsilon}\,, \qquad \text{(A.89)}$$

$$C_{21}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = -\int_0^1 x^2\, dx \int_0^x \frac{dy}{D - i\varepsilon}\,, \qquad \text{(A.90)}$$

$$C_{22}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = -\int_0^1 dx \int_0^x \frac{(x-y)^2 dy}{D - i\varepsilon}\,, \qquad \text{(A.91)}$$

$$C_{23}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = -\int_0^1 x\, dx \int_0^x \frac{(x-y)dy}{D - i\varepsilon}\,, \qquad \text{(A.92)}$$

$$C_{24}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = \frac{\Delta}{4} - \frac{1}{2}\int_0^1 dx \int_0^x dy\, \ln\left(\frac{D - i\varepsilon}{\mu^2}\right)\,, \qquad \text{(A.93)}$$

where

$$D \equiv p^2x^2 + p_2^2y^2 + (p_1^2 - p_2^2 - p^2)xy + (m_c^2 - m_a^2 - p^2)x$$
$$+(m_b^2 - m_c^2 + p^2 - p_1^2)y + m_a^2\,. \tag{A.94}$$

In addition, the following expression for C_{24} is useful and worthy of display,

$$C_{24}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = \tfrac{1}{4}\big[B_0(p_2^2; m_b^2, m_c^2) + (p_1^2 + m_a^2 - m_b^2)C_{11}$$
$$+(p^2 - p_1^2 + m_b^2 - m_c^2)C_{12} + 2m_a^2C_0 + 1\big]\,, \tag{A.95}$$

where the suppressed arguments of C_0, C_{11} and C_{12} are the same as those of C_{24}. Indeed, it is possible to express all the C_{ij} (and their derivatives) in terms of A_0, B_0 and C_0 following the same partial fractioning strategy that was used to obtain B_1 in terms of B_0 and A_0.

The integral given by eq. (A.87) can be explicitly evaluated. The resulting expression, which involves logarithms and dilogarithms, is given in Ref. [**?**], although it is not particularly illuminating in the most general case. However, one can derive a useful set of expressions in the limit of $p_1^2 = p_2^2 = p^2 = 0$. For example,

$$\begin{aligned}C_0(0,0,0; m_a^2, m_b^2, m_c^2) &= \frac{1}{m_b^2 - m_c^2}\big[B_0(0; m_a^2, m_b^2) - B_0(0; m_a^2, m_c^2)\big]\\ &= \frac{m_a^2m_b^2\ln\left(\dfrac{m_a^2}{m_b^2}\right) + m_b^2m_c^2\ln\left(\dfrac{m_b^2}{m_c^2}\right) + m_c^2m_a^2\ln\left(\dfrac{m_c^2}{m_a^2}\right)}{(m_a^2 - m_b^2)(m_b^2 - m_c^2)(m_c^2 - m_a^2)}\,,\end{aligned} \tag{A.96}$$

$$C_{24}(0,0,0; m_a^2, m_b^2, m_c^2) = \tfrac{1}{4}\big[B_0(0; m_b^2, m_c^2) + m_a^2C_0(0,0,0; m_a^2, m_b^2, m_c^2) + \tfrac{1}{2}\big]\,. \tag{A.97}$$

If two or three of the masses are degenerate, then it follows that

$$C_0(0,0,0; m^2, m^2, m_c^2) = -\frac{1}{m^2 - m_c^2}\left[1 + \frac{m_c^2}{m^2 - m_c^2}\ln\left(\frac{m_c^2}{m^2}\right)\right]\,, \tag{A.98}$$

$$C_0(0,0,0; m^2, m^2, m^2) = -\frac{1}{2m^2}\,. \tag{A.99}$$

One other limiting case is noteworthy. The following loop function arises in the calculation of the one-loop amplitude for the decay of a neutral Higgs boson to $Z\gamma$,

$$\begin{aligned}C_0(p_1^2, 0, p^2; m^2, m^2, m^2) &= \frac{1}{p^2 - p_1^2}\int_0^1 \frac{dx}{x}\,\ln\left(\frac{m^2 - x(1-x)p^2 - i\varepsilon}{m^2 - x(1-x)p_1^2 - i\varepsilon}\right)\\ &= \frac{1}{p^2 - p_1^2}\big[G(p^2/m^2) - G(p_1^2/m^2)\big]\,.\end{aligned} \tag{A.100}$$

The function $G(z)$ can be explicitly evaluated,

$$\begin{aligned}G(z) &\equiv \int_0^1 \frac{dx}{x}\,\ln\big[1 - zx(1-x) - i\varepsilon\big]\\ &= \begin{cases} 2\ln^2\left(\frac{\sqrt{-z}}{2} + \sqrt{1 - \frac{z}{4}}\right)\,, & \text{for } z < 0\,,\\ -2\big[\arcsin\big(\tfrac{1}{2}\sqrt{z}\,\big)\big]^2\,, & \text{for } 0 \le z \le 4\,,\\ -2\left[\frac{\pi}{2} + i\ln\left(\frac{\sqrt{z}}{2} + \sqrt{\frac{z}{4} - 1}\right)\right]^2\,, & \text{for } z > 4\,,\end{cases}\end{aligned} \tag{A.101}$$

where $0 \leq \arcsin\bigl(\frac{1}{2}\sqrt{z}\bigr) \leq \frac{1}{2}\pi$ (for $0 \leq z \leq 4$) employs the principal range of the real arcsine function. In the limit of $p_1^2 = p^2$, we obtain

$$C_0(p^2, 0, p^2; m^2, m^2, m^2) = \frac{1}{m^2}\left(\frac{\partial G}{\partial z}\right)_{z=p^2/m^2} = -\int_0^1 \frac{(1-x)dx}{m^2 - p^2 x(1-x) - i\varepsilon}$$
$$= \frac{1}{p^2 - 4m^2}\left[\frac{A_0(m^2)}{m^2} - B_0(p^2; m^2, m^2) + 1\right], \qquad \text{(A.102)}$$

after using eqs. (A.52) and (A.53). In the limit of $p^2 = p_1^2 = 0$, we recover eq. (A.99).

The C_{ij} functions can be expressed in terms of A_0, B_0, B_1 and C_0 via the following relations,

$$C_{11} = \frac{4(p_1\!\cdot\! p_2 R_2 - p_2^2 R_1)}{\lambda(p^2, p_1^2, p_2^2)}\,, \qquad C_{12} = \frac{4(p_1\!\cdot\! p_2 R_1 - p_1^2 R_2)}{\lambda(p^2, p_1^2, p_2^2)}\,, \qquad \text{(A.103)}$$

$$C_{21} = \frac{4(p_1\!\cdot\! p_2 R_5 - p_2^2 R_3)}{\lambda(p^2, p_1^2, p_2^2)}\,, \qquad C_{22} = \frac{4(p_1\!\cdot\! p_2 R_4 - p_1^2 R_6)}{\lambda(p^2, p_1^2, p_2^2)}\,, \qquad \text{(A.104)}$$

$$C_{23} = \frac{4(p_1\!\cdot\! p_2 R_3 - p_1^2 R_5)}{\lambda(p^2, p_1^2, p_2^2)} = \frac{4(p_1\!\cdot\! p_2 R_6 - p_2^2 R_4)}{\lambda(p^2, p_1^2, p_2^2)}\,, \qquad \text{(A.105)}$$

where

$$\lambda(p^2, p_1^2, p_2^2) \equiv 4\bigl[(p_1\!\cdot\! p_2)^2 - p_1^2 p_2^2\bigr]\,, \qquad \text{(A.106)}$$

with $p = -p_1 - p_2$ (which implies that $p_1\!\cdot\! p_2 = p^2 - p_1^2 - p_2^2$), and the R_i are defined as follows,

$$\begin{aligned}
R_1 &\equiv \tfrac{1}{2}\left[B_0\left(p^2; m_a^2, m_c^2\right) - B_0\left(p_2^2; m_b^2, m_c^2\right) - \left(p_1^2 + m_a^2 - m_b^2\right) C_0\right],\\
R_2 &\equiv \tfrac{1}{2}\left[B_0\left(p_1^2; m_a^2, m_b^2\right) - B_0\left(p^2; m_a^2, m_c^2\right) + \left(p_1^2 - p^2 - m_b^2 + m_c^2\right) C_0\right],\\
R_3 &\equiv -C_{24} - \tfrac{1}{2}\left[(p_1^2 + m_a^2 - m_b^2)C_{11} - B_1\left(p^2; m_a^2, m_c^2\right) - B_0\left(p_2^2; m_b^2, m_c^2\right)\right],\\
R_4 &\equiv -\tfrac{1}{2}\left[(p_1^2 + m_a^2 - m_b^2)C_{12} - B_1\left(p^2; m_a^2, m_c^2\right) + B_1\left(p_2^2; m_b^2, m_c^2\right)\right],\\
R_5 &\equiv -\tfrac{1}{2}\left[(p^2 - p_1^2 + m_b^2 - m_c^2)C_{11} - B_1\left(p_1^2; m_a^2, m_b^2\right) + B_1\left(p^2; m_a^2, m_c^2\right)\right],\\
R_6 &\equiv -C_{24} - \tfrac{1}{2}\left[(p^2 - p_1^2 + m_b^2 - m_c^2)C_{12} + B_1\left(p^2; m_a^2, m_c^2\right)\right],
\end{aligned}$$

where C_{24} is given by eq. (A.95) and the suppressed arguments of C_0 and the C_{ij} above are $(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2)$.

Finally, we record a number of useful symmetry properties of the C-functions when their arguments are permuted. First,

$$\begin{aligned}
C_0\left(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2\right) &= C_0\left(p_2^2, p_1^2, p^2; m_c^2, m_b^2, m_a^2\right)\\
&= C_0\left(p^2, p_1^2, p_2^2; m_c^2, m_a^2, m_b^2\right) = C_0\left(p_1^2, p^2, p_2^2; m_b^2, m_a^2, m_c^2\right)\\
&= C_0\left(p_2^2, p^2, p_1^2; m_b^2, m_c^2, m_a^2\right) = C_0\left(p^2, p_2^2, p_1^2; m_a^2, m_c^2, m_b^2\right). \qquad \text{(A.107)}
\end{aligned}$$

The same symmetry properties are also satisfied by C_{24},

$$\begin{aligned}
C_{24}\left(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2\right) &= C_{24}\left(p_2^2, p_1^2, p^2; m_c^2, m_b^2, m_a^2\right)\\
&= C_{24}\left(p^2, p_1^2, p_2^2; m_c^2, m_a^2, m_b^2\right) = C_{24}\left(p_1^2, p^2, p_2^2; m_b^2, m_a^2, m_c^2\right)\\
&= C_{24}\left(p_2^2, p^2, p_1^2; m_b^2, m_c^2, m_a^2\right) = C_{24}\left(p^2, p_2^2, p_1^2; m_a^2, m_c^2, m_b^2\right). \qquad \text{(A.108)}
\end{aligned}$$

For completeness, we list below the symmetry properties of the other C_{ij} functions, which were given in Ref. [**?**].

$$\begin{aligned}
C_{11}\left(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2\right) &= \left[-C_{12} - C_0\right]\left(p_2^2, p_1^2, p^2; m_c^2, m_b^2, m_a^2\right) \\
&= \left[-C_{11} + C_{12} - C_0\right]\left(p^2, p_1^2, p_2^2; m_c^2, m_a^2, m_b^2\right) \\
&= \left[-C_{11} + C_{12} - C_0\right]\left(p_1^2, p^2, p_2^2; m_b^2, m_a^2, m_c^2\right) \\
&= \left[-C_{12} - C_0\right]\left(p_2^2, p^2, p_1^2; m_b^2, m_c^2, m_a^2\right) \\
&= C_{11}\left(p^2, p_2^2, p_1^2; m_a^2, m_c^2, m_b^2\right) \ .
\end{aligned} \tag{A.109}$$

$$\begin{aligned}
C_{12}\left(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2\right) &= \left[-C_{11} - C_0\right]\left(p_2^2, p_1^2, p^2; m_c^2, m_b^2, m_a^2\right) \\
&= \left[-C_{11} - C_0\right]\left(p^2, p_1^2, p_2^2; m_c^2, m_a^2, m_b^2\right) \\
&= C_{12}\left(p_1^2, p^2, p_2^2; m_b^2, m_a^2, m_c^2\right) \\
&= \left[C_{11} - C_{12}\right]\left(p_2^2, p^2, p_1^2; m_b^2, m_c^2, m_a^2\right) \\
&= \left[C_{11} - C_{12}\right]\left(p^2, p_2^2, p_1^2; m_a^2, m_c^2, m_b^2\right) \ .
\end{aligned} \tag{A.110}$$

$$\begin{aligned}
C_{21}\left(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2\right) &= \left[C_{22} + 2C_{12} + C_0\right]\left(p_2^2, p_1^2, p^2; m_c^2, m_b^2, m_a^2\right) \\
&= \left[C_{21} + C_{22} - 2C_{23} + 2C_{11} - 2C_{12} + C_0\right]\left(p^2, p_1^2, p_2^2; m_c^2, m_a^2, m_b^2\right) \\
&= \left[C_{21} + C_{22} - 2C_{23} + 2C_{11} - 2C_{12} + C_0\right]\left(p_1^2, p^2, p_2^2; m_b^2, m_a^2, m_c^2\right) \\
&= \left[C_{22} + 2C_{12} + C_0\right]\left(p_2^2, p^2, p_1^2; m_b^2, m_c^2, m_a^2\right) \\
&= C_{21}\left(p^2, p_2^2, p_1^2; m_a^2, m_c^2, m_b^2\right) \ .
\end{aligned} \tag{A.111}$$

$$\begin{aligned}
C_{22}\left(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2\right) &= \left[C_{21} + 2C_{11} + C_0\right]\left(p_2^2, p_1^2, p^2; m_c^2, m_b^2, m_a^2\right) \\
&= \left[C_{21} + 2C_{11} + C_0\right]\left(p^2, p_1^2, p_2^2; m_c^2, m_a^2, m_b^2\right) \\
&= C_{22}\left(p_1^2, p^2, p_2^2; m_b^2, m_a^2, m_c^2\right) \\
&= \left[C_{21} + C_{22} - 2C_{23}\right]\left(p_2^2, p^2, p_1^2; m_b^2, m_c^2, m_a^2\right) \\
&= \left[C_{21} + C_{22} - 2C_{23}\right]\left(p^2, p_2^2, p_1^2; m_a^2, m_c^2, m_b^2\right) \ .
\end{aligned} \tag{A.112}$$

$$\begin{aligned}
C_{23}\left(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2\right) &= \left[C_{23} + C_{12} + C_{11} + C_0\right]\left(p_2^2, p_1^2, p^2; m_c^2, m_b^2, m_a^2\right) \\
&= \left[C_{21} - C_{23} + 2C_{11} - C_{12} + C_0\right]\left(p^2, p_1^2, p_2^2; m_c^2, m_a^2, m_b^2\right) \\
&= \left[C_{22} - C_{23} - C_{12}\right]\left(p_1^2, p^2, p_2^2; m_b^2, m_a^2, m_c^2\right) \\
&= \left[C_{22} - C_{23} - C_{11} + C_{12}\right]\left(p_2^2, p^2, p_1^2; m_b^2, m_c^2, m_a^2\right) \\
&= \left[C_{21} - C_{23}\right]\left(p^2, p_2^2, p_1^2; m_a^2, m_c^2, m_b^2\right) \ .
\end{aligned} \tag{A.113}$$

The following three identities are noteworthy,

$$C_{11}(p^2, q^2, p^2; M^2, \tilde{m}^2, \tilde{m}^2) = 2C_{12}(p^2, q^2, p^2; M^2, \tilde{m}^2, \tilde{m}^2)\,, \tag{A.114}$$

$$C_{21}(p^2, q^2, p^2; M^2, \tilde{m}^2, \tilde{m}^2) = 2C_{23}(p^2, q^2, p^2; M^2, \tilde{m}^2, \tilde{m}^2)\,, \tag{A.115}$$

$$\begin{aligned}
2C_{24}(p^2, q^2, p^2; M^2, \tilde{m}^2, \tilde{m}^2) &= -B_1(p^2; M^2, \tilde{m}^2) \\
&\quad + q^2\big[C_{23}(p^2, q^2, p^2; M^2, \tilde{m}^2, \tilde{m}^2) - 2C_{22}(p^2, q^2, p^2; M^2, \tilde{m}^2, \tilde{m}^2)\big].
\end{aligned} \tag{A.116}$$

Note that eqs. (A.114) and (A.115) can also be obtained by employing the last equality of eqs. (A.110) and (A.113), respectively.

Setting $q^2=0$ in eq. (A.115) yields,

$$2C_{24}(p^2,0,p^2;M^2,m^2,m^2)+B_1(p^2;M^2,m^2)=0\,. \tag{A.117}$$

This result can be established directly from the integral representations given in eqs. (A.24) and (A.93).

Using the results of eqs. (A.116) and (A.117),

$$\left(\frac{\partial C_{24}(p^2,q^2,p^2;M^2,\tilde{m}^2,\tilde{m}^2)}{\partial q^2}\right)_{q^2=0}=\tfrac{1}{2}C_{23}(p^2,0,p^2;M^2,\tilde{m}^2,\tilde{m}^2)-C_{22}(p^2,0,p^2;M^2,\tilde{m}^2,\tilde{m}^2). \tag{A.118}$$

One can check the above result by employing the integral representations given in eqs. (A.91)–(A.93). In particular,

$$\left(\frac{\partial C_{24}(p^2,q^2,p^2;M^2,\tilde{m}^2,\tilde{m}^2)}{\partial q^2}\right)_{q^2=0}=\frac{1}{12}\int_0^1\frac{x^3\,dx}{p^2x^2+(\tilde{m}^2-M^2-p^2)x+M^2}\,. \tag{A.119}$$

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