Parameterization of real orthogonal antisymmetric matrices

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Abstract

In response to a question posed by João P. Silva, I demonstrate that an arbitrary $2n\times 2n$ real orthogonal antisymmetric matrix can be parameterized by $n(n-1)$ continuous angular parameters. An algorithm is provided for constructing the general form for such a matrix. This algorithm is based on observation that $\mathrm{Sp}(n,\mathbb{R}) \cap \mathrm{O}(2n) \cong \mathrm{U}(n)$ and employs the parameterization of the coset space $\mathrm{SO}(2n)/\mathrm{U}(n)$. An explicit parameterization for a 4×4 real orthogonal antisymmetric matrix is exhibited.

1 Decomposition of a real orthogonal antisymmetric matrix

In these notes, I shall discuss the parameterization of an arbitrary real orthogonal antisymmetric matrix M, which satisfies

$$M^{\mathsf{T}} = -M\,, \qquad MM^{\mathsf{T}} = \mathbf{I}\,, \tag{1}$$

where $\mathbf{I}$ is the identity matrix.

First, we note that M is a $2n\times 2n$ nonsingular matrix such that $\det M = 1$, where n can be any positive integer. Since $M^{\mathsf{T}}M = \mathbf{I}$, it follows that $\det M = \pm 1$, which implies that that M is nonsingular. Hence, M is an even-dimensional matrix, since any odd-dimensional antisymmetric matrix M satisfies $\det M = 0$.[1] Moreover, for any even-dimensional $2n\times 2n$ antisymmetric matrix M, the *pfaffian* of M, denoted by $\mathrm{pf}\,M$, is defined by

$$\mathrm{pf}\,M = \frac{1}{2^n n!}\,\epsilon_{i_1 j_1 i_2 j_2\cdots i_n j_n} M_{i_1 j_1} M_{i_2 j_2}\cdots M_{i_n j_n}\,, \tag{2}$$

where ϵ is the rank-$2n$ Levi-Civita tensor, and the sum over repeated indices is implied. A well-known result states that for any antisymmetric matrix M,[2]

$$\det M = [\mathrm{pf}\ M]^2. \tag{3}$$

In particular, if M is also orthogonal then $\det M = 1$, in which case $\mathrm{pf}\ M = \pm 1$.

Next, we note that the eigenvalues of any real antisymmetric matrix M are purely imaginary. Moreover if λ is an eigenvalue of M then λ^* is also an eigenvalue (see, e.g., Ref. [2]). Thus, the

[1] Let M be a $d\times d$ antisymmetric matrix. Since $\det\,M = \det\,(-M^{\mathsf{T}}) = \det\,(-M) = (-1)^d \det\,M$, it follows that $\det\,M = 0$ if d is odd.

[2] For a discussion of the properties of the pfaffian, see, e.g., Ref. [1].

eigenvalues of a $2n \times 2n$ antisymmetric matrix M can be denoted by $\pm im_i$, $(i = 1, 2, \ldots, n)$ where the m_i are real and positive. We now exploit the real normal form of a nonsingular $2n \times 2n$ real antisymmetric matrix M (see, e.g., Appendix D.4 of Ref. [3]). In particular, there exists a real orthogonal matrix Q such that

$$Q^{\mathsf{T}} M Q = N \equiv \operatorname{diag}\left\{ \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & m_n \\ -m_n & 0 \end{pmatrix} \right\}, \tag{4}$$

where N is written in block diagonal form with 2×2 matrices appearing along the diagonal and the m_i are real and positive. N is called the *real normal form* of M. Note that the m_i are the positive square roots of the eigenvalues of $M^{\mathsf{T}} M$.

If in addition, M is a real orthogonal matrix, then we may use the fact that the eigenvalues of a real orthogonal matrix are complex numbers of unit modulus. In light of the above results, it follows that $m_i = 1$ for all $i = 1, 2, \ldots, n$. Hence,

$$Q^{\mathsf{T}} M Q = J \equiv \operatorname{diag}\underbrace{\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}}_{n}. \tag{5}$$

Hence, we conclude that any real orthogonal antisymmetric $2n \times 2n$ matrix M can be parameterized by

$$M = QJQ^{\mathsf{T}}, \tag{6}$$

where J is defined in eq. (5) and Q is a real orthogonal matrix. we now employ the well-known property of the pfaffian that $\operatorname{pf}(QJQ^{\mathsf{T}}) = \operatorname{pf}\ J \det Q$. In light of $\operatorname{pf}\ J = 1$, it follows that

$$\det Q = \operatorname{pf}\ M, \tag{7}$$

which determines the sign of $\det Q$.

2 Counting parameters

As discussed in Appendix D.4 of Ref. [3], the orthogonal matrix Q in eq. (6) is unique up to multiplication on the right by a $2n \times 2n$ real orthogonal matrix S that satisfies $SJS^{\mathsf{T}} = J$. Such a matrix S is an element of $\mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{O}(2n) \cong \mathrm{U}(n)$, where a proof of this isomorphism is given in Ref. [4].[3] Since $\mathrm{O}(2n)$ is parameterized by $n(2n-1)$ continuous parameters and $\mathrm{U}(n)$ is parameterized by n^2 parameters, we can use the freedom to multiply Q on the right by S to remove n^2 parameters from Q. This leaves $n(2n-1) - n^2 = n(n-1)$ parameters in Q that cannot be removed.

That is, a real orthogonal antisymmetric $2n \times 2n$ matrix M can be parameterized by $n(n-1)$ continuous parameters.

[3]See problem 1.12 on p. 41 and its solution on p. 306 of Ref. [4]. The proof of this result consists of representing an arbitrary complex unitary $n \times n$ matrix as a real $2n \times 2n$ matrix. Following Section 1.6 of Ref. [4], the corresponding real $2n \times 2n$ matrix can be identified by U_R given in eq. (13). Indeed, one can check that U_R is a $2n \times 2n$ orthogonal symplectic matrix [cf. eq. (15) and Appendix B], which exhibits the quoted isomorphism.

3 An explicit parameterization of a real orthogonal antisymmetric matrix

To find an explicit parameterization of M, one must obtain an explicit representation for Q in which the n^2 parameters represented by the matrix S have been successfully removed. We can accomplish this task as follows. First, we introduce the permuted version of J, which is the $2n \times 2n$ real orthogonal antisymmetric matrix,

$$\hat{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}, \tag{8}$$

which is related to J by

$$J = P\hat{J}P^{\mathsf{T}}, \tag{9}$$

where $\hat{P}$ is a real orthogonal permutation matrix. That is, $P\hat{J}P^{\mathsf{T}}$ performs elementary row operations that interchange pairs of rows in order to produce J. We can now define a new real orthogonal matrix $\hat{Q} = QP$ in eq. (6) to obtain

$$M = \hat{Q}\hat{J}\hat{Q}^{\mathsf{T}}. \tag{10}$$

Two cases must be considered separately depending on whether $\det \hat{Q} = +1$ or -1. Indeed, if we compute the pfaffian of M using $\mathrm{pf}(\hat{Q}\hat{J}\hat{Q}^{\mathsf{T}}) = \mathrm{pf}\ \hat{J} \det \hat{Q}$ and $\mathrm{pf}\ \hat{J} = (-1)^{n(n-1)/2}$ (see, e.g., Ref. [1]), it follows that

$$\det \hat{Q} = (-1)^{n(n-1)/2}\ \mathrm{pf}\ M, \tag{11}$$

which determines the sign of $\det \hat{Q}$.

Consider first the case of $\det \hat{Q} = +1$, which means that $\hat{Q} \in \mathrm{SO}(2n)$. Using the results of the Appendices, we see that one can always express an $\mathrm{SO}(2n)$ matrix in the following form

$$\hat{Q} = Q_c U_R, \tag{12}$$

where eqs. (68) and (79) yield

$$Q_c = \exp\begin{pmatrix} C & D \\ D & -C \end{pmatrix}, \qquad U_R = \begin{pmatrix} \mathrm{Re}\, U & -\mathrm{Im}\, U \\ \mathrm{Im}\, U & \mathrm{Re}\, U \end{pmatrix}, \tag{13}$$

in block matrix form. Here, C and D are arbitrary $n \times n$ real antisymmetric matrices, and U is an arbitrary $n \times n$ unitary matrix. In particular Q_c and U_R are both real orthogonal $2n \times 2n$ matrices. Moreover, U_R is symplectic since

$$U_R\hat{J} = \hat{J}U_R, \tag{14}$$

as is easily checked. It then follows that

$$U_R^{\mathsf{T}}\hat{J}U(R) = \hat{J}, \tag{15}$$

which is the defining property of a symplectic matrix.[4] As discussed in Appendix B, the matrices U_R provide a $2n$-dimensional orthogonal representation of the group $\mathrm{U}(n)$. Moreover, the

[4]The defining property of a symplectic matrix can employ either J or $\hat{J}$ or indeed any antisymmetric matrix related to J by conjugation [cf. eq. (9)]. See, e.g. Chapter 4, Section 29 of Ref. [5].

observation above that U_R is a real orthogonal symplectic matrix provides the demonstration that $\mathrm{Sp}(n,\mathbb{R}) \cap \mathrm{O}(2n) \cong \mathrm{U}(n)$, as previously claimed.

Finally, inserting eq. (12) into eq. (10) and making use of eq. (14), we end up with

$$M = Q_c \hat{J} Q_c^{\mathsf{T}} \,, \tag{16}$$

where

$$Q_c = \exp\begin{pmatrix} C & D \\ D & -C \end{pmatrix} \,, \tag{17}$$

and C and D are arbitrary $n \times n$ real antisymmetric matrices. We can now count parameters. Since a real antisymmetric matrix is described by $\frac{1}{2}n(n-1)$ continuous parameters, and Q_c is defined in terms of two real antisymmetric matrices, it follows that Q_c and thus M is determined by $n(n-1)$ continuous parameters, which confirms our previous claim. Finally, to obtain an explicit parameterization of M, one must evaluate the matrix exponential Q_c.

If $\det \hat{Q} = -1$, consider first the case of odd n. Then, we can introduce the $2n \times 2n$ matrix

$$\Sigma_n \equiv \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \,, \tag{18}$$

with the property that $\det \Sigma_n = (-1)^n = -1$ for odd n. In this case, we observe that

$$\Sigma_n \hat{J} \Sigma_n^{\mathsf{T}} = -\hat{J} \,, \tag{19}$$

which implies that there exists an $\mathrm{SO}(2n)$ matrix $\hat{Q}' = \hat{Q}\Sigma_n$ such that[5]

$$-M = \hat{Q}' \hat{J} \hat{Q}'^{\mathsf{T}} \,. \tag{20}$$

At this point, we can use our previous analysis to parameterize the real orthogonal antisymmetric matrix, $-M$.

If n is even, eq. (20) is not useful since $\det \hat{Q} = \det \hat{Q}' = -1$. However, a slightly modified procedure can be employed that will succeed for both cases of even and odd n. In this procedure, we introduce a $2n \times 2n$ matrix, which in block diagonal form is defined by

$$\Sigma = \mathrm{diag}(\Sigma_2 \,, \underbrace{\mathbf{I_2} \,, \mathbf{I_2} \,, \ldots \,, \mathbf{I_2}}_{2n-2}) \,, \tag{21}$$

where $\mathbf{I_2}$ is the 2×2 identity matrix and $\Sigma_2 \equiv \mathrm{diag}(1\,, -1)$. Clearly, $\det \Sigma = -1$, so we can again introduce an $\mathrm{SO}(2n)$ matrix $\hat{Q}'' = \Sigma \hat{Q}$. It then follows that

$$\Sigma M \Sigma^{\mathsf{T}} = \hat{Q}'' \hat{J} \hat{Q}''^{\mathsf{T}} \,. \tag{22}$$

At this point, we can use our previous analysis to parameterize the real orthogonal antisymmetric matrix, $\Sigma M \Sigma^{\mathsf{T}}$.

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[5]One can reach the same conclusion by noting that $\mathrm{pf}(-M) = (-1)^n \,\mathrm{pf}\, M = -\mathrm{pf}\, M$ when n is odd, in light of eq. (11).

4 Parameterizing a real orthogonal antisymmetric 4×4 matrix–Take 1

Two examples are easily checked. First, if $n = 1$, then $M = \pm \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ which involves no continuous parameters. Second, if $n = 2$, then the most general real orthogonal antisymmetric 4×4 matrix M is parameterized by two continuous parameters. Indeed, we can explicitly compute the matrix exponential given in eq. (17). Writing

$$C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \tag{23}$$

it follows that $C^2 = -c^2 \mathbf{I_2}$ and $D^2 = -d^2 \mathbf{I_2}$. Hence,

$$\begin{pmatrix} C & D \\ D & -C \end{pmatrix}^2 = -(c^2 + d^2) \begin{pmatrix} \mathbf{I_2} & 0 \\ 0 & \mathbf{I_2} \end{pmatrix}. \tag{24}$$

Thus, we can compute all powers,

$$\begin{pmatrix} C & D \\ D & -C \end{pmatrix}^{2n} = (-1)^n (c^2 + d^2)^n \begin{pmatrix} \mathbf{I_2} & 0 \\ 0 & \mathbf{I_2} \end{pmatrix}, \qquad \begin{pmatrix} C & D \\ D & -C \end{pmatrix}^{2n+1} = (-1)^n (c^2 + d^2)^n \begin{pmatrix} C & D \\ D & -C \end{pmatrix}. \tag{25}$$

We can now evaluate the exponential via its Taylor series. The end result is

$$Q_c = \exp \begin{pmatrix} C & D \\ D & -C \end{pmatrix} = \mathbf{I_4} \cos\phi + \begin{pmatrix} C & D \\ D & -C \end{pmatrix} \frac{\sin\phi}{\phi}, \tag{26}$$

where $\phi \equiv (c^2 + d^2)^{1/2}$ and $\mathbf{I_4}$ is the 4×4 identity matrix. It is convenient to introduce an angle χ,

$$\sin\chi \equiv \frac{c}{\sqrt{c^2 + d^2}}, \qquad \cos\chi = \frac{d}{\sqrt{c^2 + d^2}}. \tag{27}$$

Then, we can rewrite eq. (26) in the following form,

$$Q_c = \begin{pmatrix} \cos\phi & \sin\phi\sin\chi & 0 & \sin\phi\cos\chi \\ -\sin\phi\sin\chi & \cos\phi & -\sin\phi\cos\chi & 0 \\ 0 & \sin\phi\cos\chi & \cos\phi & -\sin\phi\sin\chi \\ -\sin\phi\cos\chi & 0 & \sin\phi\sin\chi & \cos\phi \end{pmatrix}. \tag{28}$$

Finally, inserting eq. (28) into eq. (16) and introducing $\theta \equiv 2\phi$ yields,

$$M = Q_c \hat{J} Q_c^{\mathsf{T}} = \begin{pmatrix} 0 & -\sin\theta\cos\chi & \cos\theta & \sin\theta\sin\chi \\ \sin\theta\cos\chi & 0 & -\sin\theta\sin\chi & \cos\theta \\ -\cos\theta & \sin\theta\sin\chi & 0 & \sin\theta\cos\chi \\ -\sin\theta\sin\chi & -\cos\theta & -\sin\theta\cos\chi & 0 \end{pmatrix}. \tag{29}$$

It is straightforward to check that M is a real orthogonal antisymmetric matrix. Moreover pf $M = -1$, which yields $\det\hat{Q} = 1$ in light of eq. (11), as expected. Thus, eq. (29) provides the most general expression for a real orthogonal antisymmetric 4×4 matrix M with pf $M = -1$.

In the case of pf $M = 1$, eq. (11) yields $\det \hat{Q} = -1$. Hence, to obtain the parameterization of M in this case, we employ eq. (22). The end result is,

$$M = \Sigma^{\mathsf{T}} Q_c \hat{J} Q_c^{\mathsf{T}} \Sigma = \begin{pmatrix} 0 & \sin\theta\cos\chi & \cos\theta & \sin\theta\sin\chi \\ -\sin\theta\cos\chi & 0 & \sin\theta\sin\chi & -\cos\theta \\ -\cos\theta & -\sin\theta\sin\chi & 0 & \sin\theta\cos\chi \\ -\sin\theta\sin\chi & \cos\theta & -\sin\theta\cos\chi & 0 \end{pmatrix} . \tag{30}$$

Again, it is straightforward to check that M is a real orthogonal antisymmetric matrix with pf $M = 1$.

Indeed, eqs. (29) and (30) provide the most general expressions for a real orthogonal antisymmetric 4×4 matrix.

5 Parameterizing a real orthogonal antisymmetric 4×4 matrix–Take 2

The parameterization of Q_c is not unique. In this section, we explore an alternative form,

$$Q_c' = \exp\begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \exp\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} , \tag{31}$$

where C and D are arbitrary $n \times n$ real antisymmetric matrices given in eq. (23). We can use eq. (28) to obtain

$$\exp\begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{1}{2}\theta_1\right) & \sin\left(\frac{1}{2}\theta_1\right) & 0 & 0 \\ -\sin\left(\frac{1}{2}\theta_1\right) & \cos\left(\frac{1}{2}\theta_1\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{1}{2}\theta_1\right) & -\sin\left(\frac{1}{2}\theta_1\right) \\ 0 & 0 & \sin\left(\frac{1}{2}\theta_1\right) & \cos\left(\frac{1}{2}\theta_1\right) \end{pmatrix} , \tag{32}$$

$$\exp\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{1}{2}\theta_2\right) & 0 & 0 & \sin\left(\frac{1}{2}\theta_2\right) \\ 0 & \cos\left(\frac{1}{2}\theta_2\right) & -\sin\left(\frac{1}{2}\theta_2\right) & 0 \\ 0 & \sin\left(\frac{1}{2}\theta_2\right) & \cos\left(\frac{1}{2}\theta_2\right) & 0 \\ -\sin\left(\frac{1}{2}\theta_2\right) & 0 & 0 & \cos\left(\frac{1}{2}\theta_2\right) \end{pmatrix} , \tag{33}$$

where $\theta_1 \equiv 2c$ and $\theta_2 \equiv 2d$. The factors of 2 have been inserted for later convenience.

It is noteworthy that the form of Q_c' defined in eq. (31) can be expressed as a product of simple rotation matrices. First we define the 2×2 special orthogonal matrix,

$$r(\theta) \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{34}$$

We then define the 4×4 matrix $R_{ij}(\theta)$ as a matrix whose matrix elements are given by

$$[R_{ij}(\theta)]_{k\ell} = \begin{cases} \delta_{k\ell} & \text{for } k, \ell \neq i, j\,, \\ r(\theta)\,, & \text{for } k, \ell = i, j\,. \end{cases}$$

It then follows that

$$Q_c' = R_{12}^{\mathsf{T}}\big(\tfrac{1}{2}\theta_1\big)R_{34}\big(\tfrac{1}{2}\theta_1\big)R_{14}^{\mathsf{T}}\big(\tfrac{1}{2}\theta_2\big)R_{23}\big(\tfrac{1}{2}\theta_2\big)\,. \tag{35}$$

Hence, the most general 4×4 real orthogonal antisymmetric matrix M with pf $M=-1$ is given by

$$M = Q_c'\hat{J}Q_c'^T = \begin{pmatrix} 0 & -\cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 & \sin\theta_1 \\ \cos\theta_1\sin\theta_2 & 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2 \\ -\cos\theta_1\cos\theta_2 & \sin\theta_1 & 0 & \cos\theta_1\sin\theta_2 \\ -\sin\theta_1 & -\cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2 & 0 \end{pmatrix}. \tag{36}$$

Likewise, the most general 4×4 real orthogonal antisymmetric matrix M with pf $M=1$ is then given by

$$M = \Sigma^{\mathsf{T}}Q_c'\hat{J}Q_c'^T\Sigma = \begin{pmatrix} 0 & \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 & \sin\theta_1 \\ -\cos\theta_1\sin\theta_2 & 0 & \sin\theta_1 & -\cos\theta_1\cos\theta_2 \\ -\cos\theta_1\cos\theta_2 & -\sin\theta_1 & 0 & \cos\theta_1\sin\theta_2 \\ -\sin\theta_1 & \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2 & 0 \end{pmatrix}. \tag{37}$$

Although the explicit forms for M given by eqs. (36) and (37) differ in appearance from those of eqs. (29) and (30), it is straightforward to check that they are equivalent forms if one identifies

$$\cos\theta = \cos\theta_1\cos\theta_2\,, \qquad \tan\chi = \frac{\tan\theta_1}{\sin\theta_2}\,. \tag{38}$$

One can also consider a second alternative form,

$$Q_c'' = \exp\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}\exp\begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}. \tag{39}$$

With this parameterization, the most general 4×4 real orthogonal antisymmetric matrix M with pf $M=-1$ is given by

$$M = Q_c''\hat{J}Q_c''^T = \begin{pmatrix} 0 & -\sin\theta_2 & \cos\theta_1\cos\theta_2 & \sin\theta_1\cos\theta_2 \\ \sin\theta_2 & 0 & -\sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 \\ -\cos\theta_1\cos\theta_2 & \sin\theta_1\cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1\cos\theta_2 & -\cos\theta_1\cos\theta_2 & -\sin\theta_2 & 0 \end{pmatrix}. \tag{40}$$

Likewise, the most general 4×4 real orthogonal antisymmetric matrix M with pf $M=1$ is then given by

$$M = \Sigma^{\mathsf{T}}Q_c''\hat{J}Q_c''^T\Sigma = \begin{pmatrix} 0 & \sin\theta_2 & \cos\theta_1\cos\theta_2 & \sin\theta_1\cos\theta_2 \\ -\sin\theta_2 & 0 & \sin\theta_1\cos\theta_2 & -\cos\theta_1\cos\theta_2 \\ -\cos\theta_1\cos\theta_2 & -\sin\theta_1\cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_2 & 0 \end{pmatrix}. \tag{41}$$

Once again, it is straightforward to relate these results to the previous parameterizations of a 4×4 real orthogonal antisymmetric matrix.

6 An alternative approach that only partially succeeds

Explicit calculations when $n > 2$ become quite cumbersome, since one must explicitly evaluate the matrix exponential given by eq. (17). Thus, I examined another possible line of attack starting from eq. (6). Consider the initial parameterization of $M = QJQ^{\mathsf{T}}$, which depends on a real orthogonal $2n \times 2n$ matrix Q. First we define the 2×2 special orthogonal matrix,

$$r_{ij} = \begin{pmatrix} \cos\theta_{ij} & -\sin\theta_{ij} \\ \sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} . \tag{42}$$

We then define the $N \times N$ matrix R_{ij} as a matrix whose matrix elements are given by

$$[R_{ij}]_{k\ell} = \begin{cases} \delta_{k\ell} & \text{for } k, \ell \neq i, j\,, \\ r_{ij}\,, & \text{for } k, \ell = i, j\,. \end{cases}$$

Then, one possible parameterization of the $n \times n$ matrix $\mathcal{R}_N \in \mathrm{SO}(N)$, inspired by Ref. [6], is

$$\mathcal{R}_N = R_{12}R_{13}\cdots R_{1N}R_{N-1}\,, \tag{43}$$

where R_{N-1} is the $N \times N$ matrix written in block diagonal form,

$$R_{N-1} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}_{N-1} \end{pmatrix} , \tag{44}$$

and $\mathcal{R}_{N-1}$ is the $(N-1) \times (N-1)$ matrix, $\mathcal{R}_{N-1} \in \mathrm{SO}(N-1)$. This is a recursive definition that stops once $N = 2$ is reached. This means that when expanded out, the SO(N) matrix has been expressed as a product of $\frac{1}{2}N(N-1)$ rotation matrices, each one of the form $R_{k\ell}$. In fact, any choice of the ordering of the R_{ij} matrices that appear in eq. (43) corresponds to a valid parameterization of an SO(N) matrix.

Consider the case where pf $M = \det Q = 1$ [cf. eq. (7)].[6] Applying the above results to the matrix Q for the case of $N = 2n$ with a suitable choice for the ordering of the proper rotation matrices R_{ij}, we shall write

$$Q = R_p R_{12}R_{34}\cdots R_{2n-1,2n}\,, \tag{45}$$

where R_p consists of a product of the rotation matrices R_{ij} where $i < j$ and $j \neq i+1$. Since Q consists of a product of $n(2n-1)$ rotation matrices, it follows that R_p consists of a product of $n(2n-1) - n = 2n(n-1)$ rotation matrices.

The key observation is that

$$\begin{pmatrix} \cos\theta_{ij} & -\sin\theta_{ij} \\ \sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta_{ij} & \sin\theta_{ij} \\ -\sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{46}$$

It therefore follows that

$$R_{12}R_{34}\cdots R_{2n-1,2n}JR^{\mathsf{T}}_{2n-1,2n}\cdots R^{\mathsf{T}}_{34}R^{T}_{12} = J\,, \tag{47}$$

[6]In the case of pf $M = \det Q = -1$, we can consider the corresponding decomposition of the real orthogonal antisymmetric matrix $\Sigma M \Sigma^{\mathsf{T}}$, where Σ is defined in eq. (21). Since $\mathrm{pf}(\Sigma M \Sigma^{\mathsf{T}}) = \mathrm{pf}\ M \det\Sigma = -\mathrm{pf}\ M$, it follows that in this second case, $\mathrm{pf}(\Sigma M \Sigma^{\mathsf{T}}) = 1$, and the analysis that follows is applicable.

where J is defined in eq. (5). Hence, starting from eq. (6) in the case of $\det Q = 1$, we can conclude that a real orthogonal antisymmetric $2n \times 2n$ matrix M can be parameterized by

$$M = R_p J R_p^{\mathsf{T}}\,, \tag{48}$$

where R_p is the product of $2n(n-1)$ proper rotation matrices R_{ij} where $i < j$ and $j \neq i+1$. That is, M can be parameterized by $2n(n-1)$ angles θ_{ij} where $i < j$ and $j \neq i+1$. However, this is twice the number of parameters needed to parameterize M. Thus it must be possible to write

$$R_p = R_q S\,, \tag{49}$$

where S is a real orthogonal symplectic matrix (i.e., $SJS^{\mathsf{T}} = J$) that depends on $n(n-1)$ parameters, and R_p is a real orthogonal matrix that depends on the remaining $n(n-1)$ parameters. Assuming that the decomposition give by eq. (49) is always possible (which remains to be demonstrated), then the end result is

$$M = R_q J R_q^{\mathsf{T}}\,, \tag{50}$$

which would constitute a parameterization of a general $2n \times 2n$ real orthogonal antisymmetric matrix in terms of $n(n-1)$ parameters.

Appendices: The U(n) subgroup of SO($2n$)

A. Complex representations of scalar fields

Let $\Phi_i(x)$ be a set of n complex scalar fields. The scalar Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \Phi_i)^\dagger (\partial^\mu \Phi_i) - V(\Phi_i, \Phi_i^\dagger) \tag{51}$$

is assumed to be invariant under a compact symmetry group G, under which the scalar fields transform as:

$$\Phi_i \to \mathcal{U}_i{}^j \Phi_j\,, \qquad\qquad \Phi^{\dagger\, i} \to \Phi^{\dagger\, j} (\mathcal{U}^\dagger)_j{}^i\,, \tag{52}$$

where $\mathcal{U}$ is a complex representation of G. Using a well-known theorem, all complex representations of a compact group are equivalent (via a similarity transformation) to a unitary representation. Thus, without loss of generality, we may take $\mathcal{U}$ to be a unitary $n \times n$ matrix. Explicitly,

$$\mathcal{U} = \exp[-i g_a \Lambda^a \mathcal{T}^a]\,, \tag{53}$$

where the generators $\mathcal{T}^a$ are $n \times n$ hermitian matrices. The corresponding infinitesimal transformation law is

$$\delta\Phi_i(x) = -i g_a \Lambda^a (\mathcal{T}^a)_i{}^j \Phi_j(x)\,, \tag{54}$$

$$\delta\Phi^{\dagger\, i}(x) = +i g_a \Phi^{\dagger\, j}(x) \Lambda^a (\mathcal{T}^a)_j{}^i\,, \tag{55}$$

where the g_a and Λ^a are real. One can check that the scalar kinetic energy term is invariant under U(n) transformations. The scalar potential, which is not invariant in general under the full U(n) group, is invariant under G [which is a subgroup of U(n)] if

$$(\mathcal{T}^a)_i{}^j \Phi_j \frac{\partial V}{\partial \Phi_i} - (\mathcal{T}^a)_j{}^i \Phi^{\dagger j} \frac{\partial V}{\partial \Phi^{\dagger i}} = 0 \tag{56}$$

is satisfied.

There are $2n$ independent scalar degrees of freedom, corresponding to the fields Φ_i and $\Phi^{\dagger i}$. We can also express these degrees of freedom in terms of $2n$ hermitian scalar fields consisting of ϕ_{Aj} and ϕ_{Bj} ($j = 1, 2, \ldots, n$) defined by:

$$\Phi_j = \frac{1}{\sqrt{2}}(\phi_{Aj} + i\phi_{Bj}) \,, \qquad \Phi^{\dagger j} = \frac{1}{\sqrt{2}}(\phi_{Aj} - i\phi_{Bj}) \,. \tag{57}$$

It is straightforward to compute the group transformation laws for the hermitian fields ϕ_{Aj} and ϕ_{Bj}. These are conveniently expressed by introducing a $2n$-dimensional scalar multiplet:

$$\phi(x) = \begin{pmatrix} \phi_A(x) \\ \phi_B(x) \end{pmatrix} . \tag{58}$$

That is, $\phi_{Aj}(x) = \phi_j(x)$ and $\phi_{Bj}(x) = \phi_{j+n}(x)$. Then the infinitesimal form of the group transformation law for $\phi(x)$ is given by $\phi_k(x) \to \phi_k(x) + \delta\phi_k(x)$ for $k = 1, 2, \ldots, 2n$, where

$$\delta\phi_k(x) = -ig\Lambda^a (T^a)_k{}^\ell \phi_\ell(x) \,, \tag{59}$$

and

$$iT^a = \begin{pmatrix} -\operatorname{Im}\mathcal{T}^a & -\operatorname{Re}\mathcal{T}^a \\ \operatorname{Re}\mathcal{T}^a & -\operatorname{Im}\mathcal{T}^a \end{pmatrix} . \tag{60}$$

Note that $\operatorname{Re}\mathcal{T}^a$ is symmetric and $\operatorname{Im}\mathcal{T}^a$ is antisymmetric (which follow from the hermiticity of $\mathcal{T}^a$). Thus, iT^a is a real antisymmetric $2n \times 2n$ matrix, which when exponentiated yields a real orthogonal $2n$-dimensional representation of G.

B. The embedding of U(n) in SO($2n$)

Consider a scalar field theory consisting of n identical complex fields Φ_i, with a Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \Phi_i)^\dagger (\partial^\mu \Phi_i) - V(\Phi^\dagger \Phi) \,, \tag{61}$$

where the potential function V is a function of $\Phi^{\dagger i}\Phi_i$. Such a theory is invariant under the U(n) transformation $\Phi \to U\Phi$, where U is an $n \times n$ unitary matrix.

Rewrite the Lagrangian in terms of hermitian fields ϕ_{Ai} and ϕ_{Bi} defined by:

$$\Phi_j = \frac{1}{\sqrt{2}}(\phi_{Aj} + i\phi_{Bj}) \,, \qquad \Phi^{\dagger j} = \frac{1}{\sqrt{2}}(\phi_{Aj} - i\phi_{Bj}) \,, \tag{62}$$

and introduce the $2n$-dimensional hermitian scalar field:

$$\phi(x) = \begin{pmatrix} \phi_A(x) \\ \phi_B(x) \end{pmatrix} . \tag{63}$$

One can show that the Lagrangian is actually invariant under a larger symmetry group O($2n$), corresponding to the transformation $\phi \to \mathcal{O}\phi$ where $\mathcal{O}$ is a $2n \times 2n$ orthogonal matrix.

Working in the complex basis, one can show that the Lagrangian [eq. (61)] is invariant under the transformation:

$$\Phi_i \to U_i{}^j \Phi_j + \Phi^{\dagger j} (V^\dagger)_j{}^i \,, \tag{64}$$

where U and V are complex $n \times n$ matrices, provided that the following two conditions are satisfied:

$$(i) \qquad (U^\dagger U + V^\dagger V)_i{}^j = \delta_i{}^j \,, \tag{65}$$

$$(ii) \qquad V^{\mathsf{T}} U \text{ is an antisymmetric matrix} \,. \tag{66}$$

In particular, the $2n \times 2n$ matrix

$$\mathcal{Q} = \begin{pmatrix} \mathrm{Re}(U+V) & -\,\mathrm{Im}(U+V) \\ \mathrm{Im}(U-V) & \mathrm{Re}(U-V) \end{pmatrix} \tag{67}$$

is an orthogonal matrix if U and V satisfy eqs. (65) and (66). One can prove that any $2n \times 2n$ orthogonal matrix can be written in the form of eq. (67) by verifying that $\mathcal{Q}$ is determined by $n(2n-1)$ independent parameters. This is most easily done with an infinitesimal analysis.

Using the above results, it follows that if U is a unitary $n \times n$ matrix, then the $2n \times 2n$ orthogonal matrix,[7]

$$\mathcal{Q}_U = \begin{pmatrix} \mathrm{Re}\,U & -\,\mathrm{Im}\,U \\ \mathrm{Im}\,U & \mathrm{Re}\,U \end{pmatrix} \tag{68}$$

provides an embedding of the subgroup U(n) inside O($2n$). By writing

$$\mathcal{Q}_U = \exp[-ig\Lambda^a T^a] \,, \qquad\qquad U = \exp[-ig\Lambda^a \mathcal{T}^a] \,,$$

one can show that T^a is given by eq. (60) in terms of the $\mathcal{T}^a$.

Moreover, using the well-known formula for the determinant of a block-partitioned matrix:

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det\, P \, \det\, (S - RP^{-1}Q) \,, \tag{69}$$

and writing $U_R \equiv \mathrm{Re}\,U$ and $U_I \equiv \mathrm{Im}\,U$, it follows that

$$\det\, \mathcal{Q}_U = \det\, U^{\mathsf{T}} \det\, [U_R + U_I U_R^{-1} U_I] \,, \tag{70}$$

after using $\det\, U = \det\, U^{\mathsf{T}}$. Since U is unitary by assumption (since we have chosen $V = 0$ in defining $\mathcal{Q}_U$), $U^\dagger U = I$ implies that

$$U_R^{\mathsf{T}} U_R + U_I^{\mathsf{T}} U_I = I \,, \qquad\qquad U_R^{\mathsf{T}} U_I = U_I^{\mathsf{T}} U_R \,, \tag{71}$$

after separating out the real and imaginary parts. Inserting these results into eq. (70) and using eq. (71), we find:

$$\det\, \mathcal{Q}_U = \det\, [U_R^{\mathsf{T}} U_R + U_R^{\mathsf{T}} U_I U_R^{-1} U_I] = \det\, [I - U_I^{\mathsf{T}} U_I + U_I^{\mathsf{T}} U_I] = \det\, I = 1 \,. \tag{72}$$

That is, $\mathcal{Q}_U$ is an element of SO($2n$).

[7] Note that $\mathcal{Q}_U$ is also an element of Sp($n, \mathbb{R}$) in light of eq. (15). Hence, it follows that $\mathrm{Sp}(n,\mathbb{R}) \cap \mathrm{O}(2n) \cong \mathrm{U}(n)$.

C. The embedding of the $\mathfrak{u}(n)$ Lie subalgebra inside $\mathfrak{so}(2n)$

We begin with the following theorem, which is useful in the analysis of spontaneous symmetry break of an SO($2n$) symmetric potential of a theory of a second-rank antisymmetric tensor multiplet of scalars [7]. In this section, I_n (I_{2n}) is the $n \times n$ ($2n \times 2n$) identity matrix.

Theorem: Suppose that Σ_0 is a $2n \times 2n$ real antisymmetric matrix that satisfies $\Sigma_0^{\mathsf{T}}\Sigma_0 = \Sigma_0\Sigma_0^{\mathsf{T}} = c^2 I_{2n}$ for some real number c. Then, if the generators of the Lie algebra of SO($2n$), henceforth denoted by $\mathfrak{so}(2n)$, in the defining ($2n$-dimensional) representation are given by $\{T_a, X_b\}$, where the iT_a and iX_b are real antisymmetric $2n \times 2n$ matrices that satisfy:

$$T_a\Sigma_0 + \Sigma_0 T_a^{\mathsf{T}} = 0\,, \tag{73}$$

$$X_b\Sigma_0 - \Sigma_0 X_b^{\mathsf{T}} = 0\,, \tag{74}$$

then the T_a span a $\mathfrak{u}(n)$ Lie subalgebra of $\mathfrak{so}(2n)$, while the remaining generators, X_b, span elements of $\mathfrak{so}(2n)$ whose exponentials comprise the SO($2n$)/U(n) homogeneous space. Moreover, $\mathrm{Tr}(T_a X_b) = 0$.

Proof: First, I show that if $\Sigma_0^{\mathsf{T}}\Sigma_0 = \Sigma_0\Sigma_0^{\mathsf{T}} = c^2 I_{2n}$ and $T_a\Sigma_0 + \Sigma_0 T_a^{\mathsf{T}} = 0$, then the T_a span an U(n) Lie subalgebra. Note that these two conditions imply:

$$c^2\, T_a^{\mathsf{T}} = -\Sigma_0^{\mathsf{T}} T_a \Sigma_0\,. \tag{75}$$

As noted in eq. (4), for any even-dimensional real antisymmetric matrix M, there exists a real orthogonal matrix W such that $WMW^{\mathsf{T}} = \mathrm{diag}(\mathcal{J}_1\,, \mathcal{J}_2\,, \ldots\,, \mathcal{J}_n)$ is block diagonal, where each block is a 2×2 matrix of the form $\mathcal{J}_n \equiv \left(\begin{smallmatrix} 0 & z_n \\ -z_n & 0\end{smallmatrix}\right)$ where $z_n \in \mathbb{R}$ and the z_n^2 are the eigenvalues of MM^{T} (or $M^{\mathsf{T}}M$).[8] Applying this result to Σ_0, note that the eigenvalues of $\Sigma_0\Sigma_0^{\mathsf{T}}$ are all degenerate and equal to c^2. Moreover, since the matrix

$$\hat{J} \equiv \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\,, \tag{76}$$

satisfies $\hat{J}\hat{J}^{\mathsf{T}} = I_{2n}$, it follows that one can find real orthogonal matrices W_1 and W_2 such that $W_1\Sigma_0 W_1^{\mathsf{T}} = cW_2\hat{J}W_2^{\mathsf{T}} = \mathrm{diag}(c\mathcal{J}, c\mathcal{J}, \ldots, c\mathcal{J})$, where $\mathcal{J} \equiv \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$. That is, the factorizations of Σ_0 and $c\hat{J}$ yield the same block diagonal matrix consisting of n identical 2×2 blocks consisting of $c\mathcal{J}$. Thus, there exists a real orthogonal matrix $V = W_2^{-1}W_1$ such that $V\Sigma_0 V^{\mathsf{T}} = c\,\hat{J}$. The inverse of this result is $V\Sigma_0^{\mathsf{T}}V^{\mathsf{T}} = -c\,\hat{J}$ (since $\hat{J}^{\mathsf{T}} = -\hat{J}$). I now define $\widetilde{T}_a \equiv VT_aV^{\mathsf{T}}$. Then eq. (75) implies that

$$\widetilde{T}_a^{\mathsf{T}} = \frac{-1}{c^2} V\Sigma_0^{\mathsf{T}} V^{\mathsf{T}} \widetilde{T}_a V \Sigma_0 V^{\mathsf{T}} = \hat{J}\widetilde{T}_a\hat{J}\,. \tag{77}$$

Likewise, one can use the same matrix V to define $\widetilde{X}_b \equiv VX_bV^{\mathsf{T}}$. By an analogous computation, $c^2X^{\mathsf{T}} = \Sigma_0^{\mathsf{T}} X \Sigma_0$, which implies that $\widetilde{X}_b^{\mathsf{T}} = -\hat{J}\widetilde{X}_b\hat{J}$.

Recall that that T_a and X_b are both antisymmetric $2n \times 2n$ matrices. Then, $\widetilde{T}_a \equiv VT_aV^{\mathsf{T}}$ and $\widetilde{X}_a \equiv VX_aV^{\mathsf{T}}$ are also antisymmetric. Hence, it follows that

$$\widetilde{T}_a = -\hat{J}\widetilde{T}_a\hat{J}\,, \qquad \widetilde{X}_a = \hat{J}\widetilde{X}_a\hat{J}\,. \tag{78}$$

[8] In particular, it is always possible to find a suitable choice for the unitary matrix W such that the z_i are real and non-negative. For further details, see Appendix D of Ref. [3].

Using the explicit form for $\hat{J}$, eq. (78) implies that $\widetilde{T}_a$ and $\widetilde{X}_b$ take the following block form:

$$i\,\widetilde{T}_a = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\,, \qquad\qquad i\widetilde{X}_b = \begin{pmatrix} C & D \\ D & -C \end{pmatrix}\,, \tag{79}$$

where A, B, C and D are $n \times n$ real matrices such that A, C and D are antisymmetric and B is symmetric. Thus, I have exhibited a similarity transformation (note that $V^{\mathsf{T}} = V^{-1}$) that transforms the basis of the Lie algebra spanned by the T_a to one that is spanned by the $\widetilde{T}_a$. Moreover, consider the isomorphism that maps $i\,\widetilde{T}_a$ given in eq. (79) to the $n \times n$ matrix $A + iB$. Since $(A + iB)^\dagger = (A - iB)^{\mathsf{T}} = -(A + iB)$, we see that the $A + iB$ are anti-hermitian generators (which are not generally traceless) that span a $\mathfrak{u}(n)$ subalgebra of $\mathfrak{so}(2n)$. We can check the number of $\mathfrak{u}(n)$ generators by counting the number of degrees of freedom of one real antisymmetric and one real symmetric matrix: $\frac{1}{2}n(n-1) + \frac{1}{2}n(n+1) = n^2$, as expected.

Finally, multiplying the two equations $c^2\,T_a^{\mathsf{T}} = -\Sigma_0^{\mathsf{T}} T_a \Sigma_0$ and $c^2 X_b^{\mathsf{T}} = \Sigma_0^{\mathsf{T}} X_b \Sigma_0$, it follows that $c^2\,T_a^{\mathsf{T}} X_b^{\mathsf{T}} = -\Sigma_0^{\mathsf{T}} T_a X_b \Sigma_0$ (after employing $\Sigma_0^{\mathsf{T}}\Sigma_0 = c^2 I_{2n}$). Taking the trace, it follows that $\operatorname{Tr} T_a X_b = -\operatorname{Tr} T_a X_b$, and we conclude that $\operatorname{Tr} T_a X_b = 0$.

To show that the $\{T_a, X_b\}$ span the full $\mathfrak{so}(2n)$ Lie algebra, we have already noted above that there are n^2 generators, $\{T_a\}$. In addition, there are $n(n-1)$ generators, $\{X_a\}$, corresponding to the number of parameters describing two real antisymmetric matrices [see eq. (79)]. Thus, the total number of generators is $n(2n-1)$ which matches the total number of $\mathfrak{so}(2n)$ generators.

References

[1] H.E. Haber, "Notes on antisymmetric matrices and the pfaffian," available on the Web via http://scipp.ucsc.edu/~haber/webpage/pfaffian2.pdf.

[2] J. Gallier and D. Xu, "Computing exponentials of skew-symmetric matrices and logarithms of orthogonal matrices," Int. J. Robotics and Automation **17**, no. 4, 1 (2002).

[3] H.K. Dreiner, H.E. Haber and S.P. Martin, "Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry," Phys. Rept. **494**, 1 (2010) [arXiv:0812.1594 [hep-ph]].

[4] Andrew Baker, *Matrix Groups: An Introduction to Lie Group Theory* (Springer-Verlag, London, UK, 2002).

[5] Hans Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices* (P. Noordhoff Ltd., Groningen, Holland, 1950).

[6] Francis D. Murnaghan, *The Unitary and Rotation Groups* (Spartan Books, Washington, DC, 1962).

[7] L.-F. Li, *Phys. Rev.* **D9**, 1723 (1974); V. Elias, S. Eliezer and A.R. Swift, *Phys. Rev.* **D12**, 3356 (1975); D. Wu, *Nucl. Phys.* **B199**, 523 (1982) [E: **B213**, 545 (1983)].