What is the group of conjugate symplectic matrices?

Howard E. Haber

Santa Cruz Institute for Particle Physics

University of California, Santa Cruz, CA 95064, USA

June 26, 2017

Abstract

The group of complex symplectic matrices, Sp(n,$\mathbb{C}$) is defined as the set of $2n \times 2n$ complex matrices, $\{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\mathsf{T}}JM = J\}$, where $J \equiv \left(\begin{smallmatrix} 0 & \mathbf{I_n} \\ -\mathbf{I_n} & 0 \end{smallmatrix}\right)$, and $\mathbf{I_n}$ is the $n \times n$ identity matrix. If the transpose is replaced by hermitian conjugate, then one obtains the group of conjugate symplectic matrices, $\{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\dagger}JM = J\}$. In these notes, we demonstrate that the group of conjugate symplectic matrices is isomorphic to $\mathrm{U}(n,n) = \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\dagger}I_{n,n}M = I_{n,n}\}$, where $I_{n,n} \equiv \left(\begin{smallmatrix} \mathbf{I_n} & 0 \\ 0 & -\mathbf{I_n} \end{smallmatrix}\right)$.

The group of complex symplectic matrices, Sp(n,$\mathbb{C}$) is defined as the set of $2n \times 2n$ complex matrices,[1]

$$\mathrm{Sp}(n,\mathbb{C}) \equiv \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\mathsf{T}}JM = J\}\,, \tag{1}$$

where the matrix J is defined in block form by,

$$J \equiv \begin{pmatrix} 0 & \mathbf{I_n} \\ -\mathbf{I_n} & 0 \end{pmatrix}, \tag{2}$$

with 0 denoting the $n \times n$ zero matrix and $\mathbf{I_n}$ the $n \times n$ identity matrix. Sp(n,$\mathbb{C}$) is a complex simple Lie group. Moreover, the determinant of any complex symplectic matrix is 1.[2]

If the transpose is replaced by hermitian conjugate in eq. (1), then one obtains the group of conjugate symplectic matrices,

$$G = \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\dagger}JM = J\}\,, \tag{3}$$

following the terminology or Refs. [1, 2]. Indeed, the group of conjugate symplectic matrices is a Lie group that is neither semi-simple nor complex. First, note that taking the determinant of the condition $M^{\dagger}JM = J$ implies that $\det M$ is a complex phase of unit magnitude. This group of phases constitutes a normal U(1) subgroup of G, which implies that G is not semi-simple. Second, G not a complex Lie group since it is defined by a condition, $M^{\dagger}JM = J$, that is not holomorphic. Nevertheless, it should be possible to identify this group as being isomorphic to one of the real forms of $\mathrm{GL}(2n,\mathbb{C})$. In these notes, I shall identify the real Lie group in question.

[1] The reader should be aware that in the literature, the group defined by eq. (1) is often denoted by Sp($2n$,$\mathbb{C}$).

[2] Taking the determinant of the defining relation $M^{\mathsf{T}}JM = J$ yields $\det M = \pm 1$. However, since all elements of Sp(n,$\mathbb{C}$) are connected to the identity, it follows that $\det M = 1$. Another proof of this fact, which employs the properties of the pfaffian, is provided in Ref. [3].

We begin by diagonalizing the matrix J defined in eq. (2),

$$\frac{1}{2}\begin{pmatrix} \mathbf{I_n} & -i\mathbf{I_n} \\ \mathbf{I_n} & i\mathbf{I_n} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{I_n} \\ -\mathbf{I_n} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{I_n} & \mathbf{I_n} \\ i\mathbf{I_n} & -i\mathbf{I_n} \end{pmatrix} = i\begin{pmatrix} \mathbf{I_n} & 0 \\ 0 & -\mathbf{I_n} \end{pmatrix}. \tag{4}$$

It is convenient to introduce the notation for the $(p+q)\times(p+q)$ diagonal matrix,

$$I_{p,q} \equiv \mathrm{diag}(\underbrace{1,1,\ldots,1}_{p},\underbrace{-1,-1,\ldots,-1}_{q}). \tag{5}$$

In particular, note that

$$I_{n,n} \equiv \begin{pmatrix} \mathbf{I_n} & 0 \\ 0 & -\mathbf{I_n} \end{pmatrix}. \tag{6}$$

Employing the unitary $2n\times 2n$ matrix,

$$U \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I_n} & \mathbf{I_n} \\ i\mathbf{I_n} & -i\mathbf{I_n} \end{pmatrix}, \tag{7}$$

allows us to rewrite eq. (4) as

$$U^\dagger J U = iI_{n,n}\,. \tag{8}$$

We now introduce the matrix N defined by

$$N \equiv U^\dagger M U\,, \tag{9}$$

where M is a conjugate symplectic matrix defined in eq. (3). Using eq. (8) and the fact that U is unitary, it follows that the condition $M^\dagger J M = J$ yields,

$$N^\dagger I_{n,n} N = I_{n,n}\,. \tag{10}$$

We now recognize the Lie group [4],

$$\mathrm{U}(p,q) \equiv \{V \in \mathrm{GL}(n,\mathbb{C})\,|\,V^\dagger I_{p,q} V = I_{p,q}\}\,, \quad \text{where } p+q=n. \tag{11}$$

Note that the Lie groups $\mathrm{U}(p\,,n-p)$ are real forms[3] of the complex Lie group $\mathrm{GL}(n,\mathbb{C})$. In light of eq. (10), it follows that $N \in \mathrm{U}(n,n)$.

Thus, eq. (9) establishes a bijective map from the group of conjugate symplectic matrices to the Lie group $\mathrm{U}(n,\,n)$ that takes $M \longmapsto N$, which implies that the following is an isomorphism,

$$\mathrm{U}(n,n) \cong \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^\dagger J M = J\}\,. \tag{12}$$

[3] That is, the complexification of $\mathrm{U}(p\,,n-p)$, for any value of $p=0,1,2,\ldots,n$, yields the complex Lie group $\mathrm{GL}(n,\mathbb{C})$.

References

[1] A. Bunse-Gerstner, R. Byers, V. Mehrmann, "A chart of numerical methods for structured eigenvalue problems," SIAM J. Matrix Anal. Appl. 13, 419 (1992).

[2] D.S. Mackey, N. Mackey, F. Tisseur, "Structured tools for structured matrices," Electron. J. Linear Algebra 10, 106 (2003).

[3] H.E. Haber, "Notes on antisymmetric matrices and the pfaffian," available on the Web via `http://scipp.ucsc.edu/~haber/webpage/pfaffian2.pdf`.

[4] See, e.g., Table 8.1 on p. 150 of J.F. Cornwell, *Group Theory in Physics—An Introduction* (Academic Press, San Diego, CA, 1997).