Massless Majorana and Weyl fermions cannot be distinguished

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In this note, I will demonstrate that a theory of a massless neutrino (in $3+1$-dimensional quantum field theory) can be described either as a theory of a massless Weyl fermion or as a theory of a massless four-component Majorana fermion. The two formulations are indistinguishable, as they arise from exactly the same Lagrangian when expressed in terms of two-component fermions. I will also exhibit the equivalence of one massive Dirac fermion with a theory of two mass-degenerate Majorana fermions. Finally, I will argue that no discontinuities arise when taking the $m \to 0$ limit of a theory of one Majorana fermion of mass m, or when taking the $m_1 \to m_2$ limit of a theory of two Majorana fermions of mass m_1 and m_2, respectively.

First, some facts about two-component anti-commuting spinors. Introduce undotted spinors ξ_α with $\alpha = 1, 2$ and dotted spinors $\overline{\eta}^{\dot\alpha} \equiv \eta^{\alpha *}$. Indices are raised and lowered with $\epsilon^{\alpha\beta} = i\sigma^2$ (where σ^2 is the usual Pauli matrix) and $\epsilon_{\alpha\beta} = -i\sigma^2$ (ϵ-tensors with dotted indices are defined similarly). We introduce $\sigma^\mu = (I; \sigma^i)$ and $\overline{\sigma}^\mu = (I; -\sigma^i)$, where I is the 2×2 identity matrix. Explicitly, σ^μ and $\overline{\sigma}^\mu$ possess the following spinor index structure: $\sigma^\mu_{\alpha\dot\alpha}$ and $\overline{\sigma}^{\mu\dot\alpha\alpha}$. Finally, spinor products are defined as

$$\chi\xi \equiv \chi^\alpha \xi_\alpha = \epsilon^{\alpha\beta}\chi_\beta \xi_\alpha \,, \tag{1}$$

$$\overline{\chi}\overline{\xi} \equiv \overline{\chi}_{\dot\alpha}\overline{\xi}^{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\overline{\chi}^{\dot\beta}\overline{\xi}^{\dot\alpha} \,. \tag{2}$$

Note that $\chi\xi = \xi\chi$ and $\overline{\chi}\overline{\xi} = \overline{\xi}\overline{\chi}$ due to the anti-commuting properties of the spinors and the antisymmetry of the ϵ-tensor. Finally, hermitian conjugation of a spinor product reverses the order of the spinors. Using the above results, it follows that $(\chi\xi)^\dagger = \overline{\chi}\overline{\xi}$. The following notation for spinor products is also useful:

$$\chi\sigma^\mu\overline{\eta} \equiv \chi^\alpha \sigma^\mu_{\alpha\dot\alpha}\overline{\eta}^{\dot\alpha} \,, \tag{3}$$

$$\overline{\chi}\,\overline{\sigma}^\mu\eta \equiv \overline{\chi}_{\dot\alpha}\sigma^{\mu\dot\alpha\alpha}\eta_\alpha \,. \tag{4}$$

One can easily show that $(\chi\sigma^\mu\overline{\xi})^\dagger = \xi\sigma^\mu\overline{\chi}$ and $\chi\sigma^\mu\overline{\eta} = -\overline{\eta}\,\overline{\sigma}^\mu\chi$.

Four-component notation is established as follows. A four-component spinor has the form:

$$\psi = \begin{pmatrix} \xi_\alpha \\ \overline{\eta}^{\dot{\alpha}} \end{pmatrix} . \tag{5}$$

The γ-matrices can be written as:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_{\mu\alpha\dot{\beta}} \\ \overline{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix} , \tag{6}$$

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} . \tag{7}$$

Left and right-handed projection operators are:

$$P_L \equiv \tfrac{1}{2}(\mathbf{1} - \gamma_{\mathbf{5}}) , \tag{8}$$

$$P_R \equiv \tfrac{1}{2}(\mathbf{1} + \gamma_{\mathbf{5}}) , \tag{9}$$

where $\mathbf{1}$ is the 4×4 identity matrix. A Majorana spinor is obtained by setting $\eta = \xi$ in eq. (5):

$$\psi_M = \begin{pmatrix} \xi_\alpha \\ \overline{\xi}^{\dot{\alpha}} \end{pmatrix} . \tag{10}$$

If we introduce the charge conjugation matrix $C = i\gamma^0\gamma^2$, and define the charge-conjugated spinor by $\psi^c \equiv C\overline{\psi}^T$, then it is easy to check that $\psi^c_M = \psi_M$.

It is useful to develop a translation table between two-component and four-component spinors. Introducing two four-component spinors

$$\psi_1 = \begin{pmatrix} \xi_1 \\ \overline{\eta}_1 \end{pmatrix} , \qquad \psi_2 = \begin{pmatrix} \xi_2 \\ \overline{\eta}_2 \end{pmatrix} \tag{11}$$

the following translation table is obtained:

$$\overline{\psi}_1 P_L \psi_2 = \eta_1 \xi_2 , \tag{12}$$

$$\overline{\psi}_1 P_R \psi_2 = \overline{\eta}_2 \overline{\xi}_1 , \tag{13}$$

$$\overline{\psi}_1 \gamma^\mu P_L \psi_2 = \overline{\xi}_1 \overline{\sigma}^\mu \xi_2 , \tag{14}$$

$$\overline{\psi}_1 \gamma^\mu P_R \psi_2 = -\overline{\eta}_2 \overline{\sigma}^\mu \eta_1 . \tag{15}$$

I shall first write down the theory of an electron and a massless neutrino interacting via charged W exchange. (I have also obtained the entire Standard Model in two-component notation but this will not be needed here.) Then, I will convert from two-component to four-component notation to obtain the well-known form for the interaction of an electron and neutrino.

First, introduce the following two-component fields: an SU(2) doublet $\psi_L = (\psi_{L_1}, \psi_{L_2})$ with hypercharge $Y = -1$, and an SU(2) singlet ψ_E with hypercharge $Y = +2$. The corresponding electric charges are obtained as usual from $Q = T_3 + \frac{1}{2}Y$. The relevant part of the Standard Model Lagrangian (after generating mass for the electron) in two-component notation is:

$$\begin{aligned}\mathcal{L} &= i(\overline{\psi}_{L_1}\overline{\sigma}^\mu\partial_\mu\psi_{L_1} + \overline{\psi}_{L_2}\overline{\sigma}^\mu\partial_\mu\psi_{L_2} + \overline{\psi}_E\overline{\sigma}^\mu\partial_\mu\psi_E) \\ &\quad -\frac{g}{\sqrt{2}}\left[\overline{\psi}_{L_1}\overline{\sigma}^\mu\psi_{L_2}W^+_\mu + \overline{\psi}_{L_2}\overline{\sigma}^\mu\psi_{L_1}W^-_\mu\right] - m_e\left(\psi_{L_2}\psi_E + \overline{\psi}_{L_2}\overline{\psi}_E\right)\,.\end{aligned} \tag{16}$$

To convert to four-component notation, one introduces four-component spinors

$$e = \begin{pmatrix} \psi_{L_2} \\ \overline{\psi}_E \end{pmatrix}, \qquad \nu = \begin{pmatrix} \psi_{L_1} \\ 0 \end{pmatrix}. \tag{17}$$

Note that in the Standard Model (with a massless neutrino), there is no right-handed neutrino; *i.e.*, there is no two-component fermion to pair up with ψ_{L_1}. Using the translation table above, it is easy to obtain the corresponding Lagrangian in four-component notation:

$$\mathcal{L} = i(\overline{\nu}\gamma^\mu\partial_\mu\nu + \overline{e}\gamma^\mu\partial_\mu e) - \frac{g}{\sqrt{2}}\left[\overline{\nu}\gamma^\mu P_L e W^+_\mu + \overline{e}\gamma^\mu P_L \nu W^-_\mu\right] - m_e\overline{e}e\,. \tag{18}$$

In deriving this result, I used the fact that $\nu = P_L\nu$ and

$$\begin{aligned}\overline{e}\gamma^\mu\partial_\mu e &= \overline{\psi}_{L_2}\overline{\sigma}^\mu\partial_\mu\psi_{L_2} - (\partial_\mu\overline{\psi}_E)\overline{\sigma}^\mu\psi_E \\ &= \overline{\psi}_{L_2}\overline{\sigma}^\mu\partial_\mu\psi_{L_2} + \overline{\psi}_E\overline{\sigma}^\mu\partial_\mu\psi_E + \text{total divergence}\,.\end{aligned} \tag{19}$$

The total divergence can be dropped.

However, I can just as well introduce a four-component Majorana neutrino:

$$\nu_M = \begin{pmatrix} \psi_{L_1} \\ \overline{\psi}_{L_1} \end{pmatrix}. \tag{20}$$

Note that since $P_L + P_R = \mathbf{1}$, one can write

$$\begin{aligned}\tfrac{1}{2}\overline{\nu}_M\gamma^\mu\partial_\mu\nu_M &= \tfrac{1}{2}\overline{\nu}_M\gamma^\mu(P_L + P_R)\partial_\mu\nu_M \\ &= \tfrac{1}{2}\left[\overline{\psi}_{L_1}\overline{\sigma}^\mu\partial_\mu\psi_{L_1} - (\partial_\mu\overline{\psi}_{L_1})\overline{\sigma}^\mu\psi_{L_1}\right] \\ &= \overline{\psi}_{L_1}\overline{\sigma}^\mu\partial_\mu\psi_{L_1} - \tfrac{1}{2}\partial_\mu(\overline{\psi}_{L_1}\overline{\sigma}^\mu\psi_{L_1})\,.\end{aligned} \tag{21}$$

Dropping the total divergence as before, it follows that the Lagrangian in four-component notation can be written as

$$\mathcal{L} = \tfrac{1}{2} i\overline{\nu}_M \gamma^\mu \partial_\mu \nu_M + i\overline{e}\gamma^\mu \partial_\mu e - \frac{g}{\sqrt{2}} \left[\overline{\nu}_M \gamma^\mu P_L e W^+_\mu + \overline{e}\gamma^\mu P_L \nu_M W^-_\mu \right] - m_e \overline{e} e \,. \tag{22}$$

Note that $\nu = P_L \nu_M$. Thus, we see that for a massless neutrino, we can either use the Majorana neutrino or the Weyl neutrino. The two theories are indistinguishable, since they were derived from the same Lagrangian expressed in two-component notation. The factor of $\frac{1}{2}$ that appears in eq. (22) in front of the kinetic energy term for the Majorana neutrino is correct. Just like in scalar field theory, if one uses real fields, the corresponding coefficient of the kinetic energy term is $\frac{1}{2}$ while for complex fields it is 1.

Finally, it is simple to add a mass term to the theory of a Majorana neutrino. For example, in the see-saw mechanism, one introduces a new two-component field ψ_N which is completely neutral under SU(2)×U(1). In this case, one adds a kinetic energy term, $i\overline{\psi}_N \overline{\sigma}^\mu \partial_\mu \psi_N$, and the following mass term

$$-\mathcal{L}_{\rm mass} = \tfrac{1}{2} \left(\psi_{L_1} \;\; \psi_N \right) \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix} \begin{pmatrix} \psi_{L_1} \\ \psi_N \end{pmatrix} \quad + \quad \text{h.c.} \tag{23}$$

Above, m_D is a Dirac mass while M is the Majorana mass. For $M \gg m_D$, the eigenvalues of the mass matrix are approximately M and $-m_D^2/M$. One can define two new two-component fields:

$$i\psi_a \simeq \psi_{L_1} - \frac{m_D}{M}\psi_N \,, \tag{24}$$

$$\psi_b \simeq \psi_N + \frac{m_D}{M}\psi_{L_1} \,. \tag{25}$$

The extra factor of i above is required in order to obtain positive masses. (It can be showed that the sign of the two-component fermion mass is related to the corresponding CP properties of the field.) Then, it follows that

$$-\mathcal{L}_{\rm mass} = \tfrac{1}{2} \left[\frac{m_D^2}{M} \psi_a \psi_a + M \psi_b \psi_b + \text{h.c.} \right] + \mathcal{O}\left(\frac{m_D^3}{M^2} \right) \,, \tag{26}$$

which corresponds to a theory of two Majorana fermions, one very light and one very heavy.

One must employ Majorana neutrino spinors to convert the results of eq. (26) to four-component notation. For simplicity, consider first the case of one two-component massive fermion field:

$$\mathcal{L} = i\overline{\chi}\,\overline{\sigma}^\mu \partial_\mu \chi - \tfrac{1}{2} m (\chi\chi + \overline{\chi}\,\overline{\chi}) \,. \tag{27}$$

Converting to four-component notation and discarding the total divergence as before, we end up with a theory of one massive Majorana fermion

$$\mathcal{L} = \tfrac{1}{2} i\overline{\psi}_M \gamma^\mu \partial_\mu \psi_M - \tfrac{1}{2} m \overline{\psi}_M \psi_M \,, \tag{28}$$

where the four-component Majorana spinor is defined as

$$\psi_M = \begin{pmatrix} \chi \\ \overline{\chi} \end{pmatrix} , \tag{29}$$

as discussed earlier. The extension to a theory of two Majorana fermions of unequal masses is straightforward.

Finally, we note two different limits. As $M \to \infty$, we can discard the heavy Majorana neutrino from the low-energy effective theory. The mass of the light neutrino goes to zero smoothly. In the zero mass limit, we can either use a four-component Majorana or Weyl description. In the limit of $M \to 0$, the theory contains two mass-degenerate four-component Majorana neutrinos (both with mass m_D). We can express this result equivalently as a theory of one four-component Dirac neutrino with mass m_D. For an explicit verification, consider the following theory in two-component notation:

$$\mathcal{L} = i(\overline{\xi}\,\overline{\sigma}^\mu \partial_\mu \xi + \overline{\eta}\,\overline{\sigma}^\mu \partial_\mu \eta) - m(\xi\eta + \overline{\xi}\overline{\eta}) \,. \tag{30}$$

Introducing the four-component spinor

$$\psi_D = \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix} , \tag{31}$$

as before, and using the translation table, one obtains in four-component notation (after discarding a total divergence)

$$\mathcal{L} = i\overline{\psi}_D \gamma^\mu \partial_\mu \psi_D - m\overline{\psi}_D \psi_D \,. \tag{32}$$

However, one can perform the analysis differently. Before converting from two-component notation, note that eq. (30) involves the following 2×2 fermion mass matrix:

$$\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} . \tag{33}$$

The corresponding eigenvalues are $\pm m$. Introduce new two-component fields

$$\psi_a = \frac{\xi + \eta}{\sqrt{2}} \,, \tag{34}$$

$$i\psi_b = \frac{\xi - \eta}{\sqrt{2}} \,. \tag{35}$$

Again, we introduce the factor of i in order to obtain positive masses for the Majorana fermions. Rewriting eq. (30) in terms of the new fields, one obtains

$$\mathcal{L} = i(\overline{\psi}_a \overline{\sigma}^\mu \partial_\mu \psi_a + \overline{\psi}_b \overline{\sigma}^\mu \partial_\mu \psi_b) - \tfrac{1}{2} m \left(\psi_a \psi_a + \overline{\psi}_a \overline{\psi}_a + \psi_b \psi_b + \overline{\psi}_b \overline{\psi}_b \right) . \quad (36)$$

That is, the theory consists of two mass-degenerate Majorana fermions of mass m. Thus, we have explicitly demonstrated that one Dirac fermion is equivalent to two mass-degenerate Majorana fermions.

It is clear that when there is a conserved quantum number, it is more convenient to express two mass-degenerate Majorana fermions as one massive Dirac fermion. (No one would express the electron as two mass-degenerate Majorana fermions, although this can be done.) In the case of massless fermions, one likewise has a choice whether to use four-component Majorana or Weyl fermions. Again, the existence of a conserved lepton number in the theory with massless neutrinos is the reason one usually favors the Weyl over the Majorana form of the theory. Of course, in theories with massive neutrinos (assuming no unexpected mass degeneracies), one must employ Majorana fields to describe the theory. However it must be emphasized that no discontinuities arise in the limit where two Majorana fermion masses become equal or one Majorana fermion mass vanishes.