Evaluating the one-loop function arising in $h \to \gamma\gamma$

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Abstract

In these notes, we provide an explicit calculation of an integral that arises in the computation of the one-loop amplitude for the Higgs boson decay to two photons. Two additional derivations are provided in the appendices to these notes.

In the computation of the one-loop amplitude for the Higgs boson decay to two photons, the following integral arises [1–8],

$$F(z) = \lim_{\epsilon\to 0^+} F(z+i\epsilon) \equiv \lim_{\epsilon\to 0^+} \int_0^1 \frac{dx}{x} \ln\bigl[1 - zx(1-x) - i\epsilon\bigr]\,, \tag{1}$$

where z is a real parameter and ϵ is a real positive infinitesimal. The goal of this note is to provide an explicit computation of $F(z)$.

First, let us examine the range of the parameter z for which $\operatorname{Im} F(z) \neq 0$. Let us denote the argument of the logarithm in eq. (1) by the function,

$$f(x) \equiv zx^2 - zx + 1 \geq 0\,. \tag{2}$$

Observe that $\operatorname{Im} F(z) = 0$ if $f(z) > 0$ for $0 \leq x \leq 1$. In particular, $\operatorname{Im} F(z) = 0$ if $z < 4$ since the maximal value of $x(1-x)$ is $\frac{1}{4}$ over the integration range. Note that $x = \frac{1}{2}$ is a minimum of $f(x)$ if $z > 0$ and $f(\frac{1}{2}) = 1 - \frac{1}{4}z$, which implies that the minimum value of $f(x)$ at $x = \frac{1}{2}$ is negative when $z > 4$. Since $f(0) = f(1) = 1$, it follows that $f(x) < 0$ for values of x such that $0 < x_- < x < x_+ < 1$, where $x_\pm$ are the roots of $f(x)$,

$$x_\pm = \tfrac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{z}}\,\right]. \tag{3}$$

Thus,

$$\operatorname{Im} F(z) = \Theta(z-4) \int_{x_-}^{x_+} \frac{dx}{x} \operatorname{Im} \ln\bigl[1 - zx(1-x) - i\epsilon\bigr]\,, \tag{4}$$

where we have explicitly included the step function to enforce the condition that $\operatorname{Im} F(z) = 0$ if $z \leq 4$. To evaluate the imaginary part of the logarithm, we employ the principal value of the complex-valued logarithm, with the branch cut taken along the negative real axis. In particular, for any nonzero real number y and real positive infinitesimal ϵ,

$$\ln(y - i\epsilon) = \ln|y| - i\pi\Theta(-y)\,. \tag{5}$$

It then follows that $\operatorname{Im}\ln(y-i\epsilon)=-\pi\Theta(-y)$. Employing this result in eq. (4),

$$\operatorname{Im}F(z)=-\pi\Theta(z-4)\int_{x_-}^{x_+}\frac{dx}{x}=-\pi\,\Theta(z-4)\ln\left(\frac{x_+}{x_-}\right)=-\pi\,\Theta(z-4)\ln\left(\frac{1+\sqrt{1-\frac{4}{z}}}{1-\sqrt{1-\frac{4}{z}}}\right)$$
$$=-2\pi\,\Theta(z-4)\ln\left(\frac{\sqrt{z}}{2}+\sqrt{\frac{z}{4}-1}\right), \tag{6}$$

after using the explicit forms for $x_\pm$ given in eq. (3).

We now turn to the computation of $\operatorname{Re}F(z)$. We shall examine separately the cases of $z<4$ and $z>4$. In the case of $z<4$, the argument of the logarithm in eq. (1) is positive, in which case we can drop the $-i\epsilon$ term and write

$$F(z)\equiv\int_0^1\frac{dx}{x}\,\ln\bigl[1-zx(1-x)\bigr]\,,\qquad\text{for } z<4. \tag{7}$$

We proceed to examine two subcases.

First, consider the case of $z<0$. In this case, it is convenient to define,

$$z=-4\sinh^2 w\,,\qquad\text{for } 0<w<\infty. \tag{8}$$

Taking the derivative of F with respect to w,

$$\frac{dF}{dw}=\frac{d}{dw}\int_0^1\frac{dx}{x}\ln\bigl[1+4x(1-x)\sinh^2 w\bigr]=4\sinh 2w\int_0^1\frac{(1-x)dx}{1+4x(1-x)\sinh^2 w}\,. \tag{9}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1+4x(1-x)\sinh^2 w=-4\sinh^2 w(x-x_+)(x-x_-)\,,\qquad\text{where } x_\pm=\pm\frac{e^{\pm w}}{2\sinh w}\,. \tag{10}$$

Hence, it follows that

$$\frac{dF}{dw}=-\frac{\sinh 2w}{\sinh^2 w}\int_0^1\frac{(1-x)dx}{(x-x_+)(x-x_-)}=-\frac{2\cosh w}{\sinh w(x_+-x_-)}\int_0^1\left(\frac{1-x_+}{x-x_+}-\frac{1-x_-}{x-x_-}\right)dx\,. \tag{11}$$

Note that $x_+\in(1,\infty)$ and $x_-\in(-\infty,0)$. Hence, the integrands above are not singular for $0\leq x\leq 1$, and the corresponding integrals are well defined.

Using eq. (10), it follows that

$$x_+-x_-=\frac{\cosh w}{\sinh w}\,,\qquad\qquad x_++x_-=1\,. \tag{12}$$

It follows that

$$\frac{dF}{dw}=-2\int_0^1\left(\frac{x_-}{x-x_+}-\frac{x_+}{x-x_-}\right)dx=-2x_-\ln\left(\frac{x_+-1}{x_+}\right)+2x_+\ln\left(\frac{x_1-1}{x_-}\right)$$
$$=-2x_-\ln\left(-\frac{x_-}{x_+}\right)+2x_+\ln\left(-\frac{x_+}{x_-}\right)=2\ln\left(-\frac{x_+}{x_-}\right)=4w\,, \tag{13}$$

after employing eqs. (10) and (12). Using the boundary condition, $F(0)=0$, one can solve the differential equation above to obtain $F(w)=2w^2$. In light of eq. (8), $w=\operatorname{arcsinh}\frac{1}{2}\sqrt{-z}$, and we end up with

$$F(z)=2\left[\operatorname{arcsinh}\left(\tfrac{1}{2}\sqrt{-z}\right)\right]^2\,, \qquad \text{for } z\leq 0. \tag{14}$$

Using the identity,

$$\operatorname{arcsinh} x=\ln\left(x+\sqrt{1+x^2}\right)\,, \tag{15}$$

one can obtain another form of eq. (14), which we can write in two different ways,

$$F(z)=2\ln^2\left(\frac{\sqrt{-z}}{2}+\sqrt{1-\frac{z}{4}}\right)=\frac{1}{2}\ln^2\left(\frac{\sqrt{1-\frac{4}{z}}+1}{\sqrt{1-\frac{4}{z}}-1}\right)\,, \qquad \text{for } z\leq 0. \tag{16}$$

Next, consider the case of $0\leq z<4$. In this case, it is convenient to define,

$$z=4\sin^2\theta\,, \qquad \text{for } 0<\theta\leq\tfrac{1}{2}\pi. \tag{17}$$

Taking the derivative of F with respect to θ,

$$\frac{dF}{d\theta}=\frac{d}{d\theta}\int_0^1\frac{dx}{x}\,\ln\left[1-4x(1-x)\sin^2\theta\right]=-4\sin 2\theta\int_0^1\frac{(1-x)dx}{1-4x(1-x)\sin^2\theta}\,. \tag{18}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1-4x(1-x)\sin^2\theta=4\sin^2\theta(x-x_+)(x-x_-)\,, \qquad \text{where } x_\pm\equiv\pm\frac{ie^{\mp i\theta}}{2\sin\theta}. \tag{19}$$

Hence, it follows that

$$\frac{dF}{d\theta}=-\frac{\sin 2\theta}{\sin^2\theta}\int_0^1\frac{(1-x)dx}{(x-x_+)(x-x_-)}=-\frac{2\cos\theta}{(x_+-x_-)\sin\theta}\int_0^1\left(\frac{1-x_+}{x-x_+}-\frac{1-x_-}{x-x_-}\right)dx\,. \tag{20}$$

Using eq. (19), it follows that

$$x_+-x_-=\frac{i\cos\theta}{\sin\theta}\,, \qquad x_++x_-=1\,. \tag{21}$$

Moreover,

$$\int_0^1\frac{(1-x_-)dx}{x-x_-}=\int_0^1\frac{x_+\,dx}{x-x_-}=\int_0^1\frac{x_+\,dx}{x-1+x_+}=-\int_0^1\frac{x_+\,dx}{x-x_+}\,, \tag{22}$$

after changing the integration variable $x\to 1-x$ in the final step above. In light of these last two results, eq. (20) yields,

$$\frac{dF}{d\theta}=2i\int_0^1\frac{dx}{x-x_+}=2i\ln\left(\frac{1-x_+}{-x_+}\right)=2i\ln\left(\frac{x_-}{-x_+}\right)=2i\ln\left(e^{2i\theta}\right)\,, \tag{23}$$

after using eq. (19) for $x_\pm$ to obtain the final result.

To complete our analysis, recall that the principal value of the complex logarithm is given by,

$$\ln z = \ln|z| + i\arg z\,, \tag{24}$$

where the principal value of the argument function is defined such that $-\pi < \arg z \leq \pi$. Since $0 < \theta \leq \frac{1}{2}\pi$ [cf. eq. (17)], it follows that $\ln(e^{2i\theta}) = 2i\theta$. Hence, eq. (23) yields,

$$\frac{dF}{d\theta} = -4\theta\,, \qquad \text{for } 0 < \theta \leq \tfrac{1}{2}\pi. \tag{25}$$

Setting $z = 0$ in eq. (7) and noting that $z = 0$ implies that $\theta = 0$, it follows that $F(\theta = 0) = 0$, which serves as an initial condition for eq. (25). Integrating eq. (25) subject to this initial condition yields,

$$F(\theta) = -2\theta^2\,, \qquad \text{for } 0 < \theta \leq \tfrac{1}{2}\pi. \tag{26}$$

From eq. (17), $\sin\theta = \frac{1}{2}\sqrt{z}$. Hence,

$$\theta = \arcsin\left(\tfrac{1}{2}\sqrt{z}\right)\,. \tag{27}$$

Plugging this result back into eq. (26) yields our final result,[1]

$$F(z) = -2\left[\arcsin\left(\tfrac{1}{2}\sqrt{z}\right)\right]^2\,, \qquad \text{for } 0 \leq z < 4\,, \tag{28}$$

where $0 \leq \arcsin\left(\frac{1}{2}\sqrt{z}\right) < \frac{1}{2}\pi$. Note that eqs. (14) and (28) are analytic continuations of each other, in light of the relation, $\arcsin(ix) = i\,\text{arcsinh}\,x$. In order to analytically continue eq. (28) into the region of $z > 4$, the following equivalent form for eq. (28) is useful,

$$F(z) = -2\left[\frac{\pi}{2} - \arccos\left(\tfrac{1}{2}\sqrt{z}\right)\right]^2\,, \qquad \text{for } 0 \leq z < 4\,. \tag{29}$$

The case of $z = 4$ can be treated separately. In this case, $1 - 4x(1-x) = (1-2x)^2$, in which case we can again drop the $-i\epsilon$ term in eq. (1). It then follows that

$$F(z=4) = \int_0^1 \frac{dx}{x}\,\ln\left[(1-2x)^2\right] = 2\int_0^{1/2}\frac{dx}{x}\,\ln(1-2x) + 2\int_{1/2}^1 \frac{dx}{x}\,\ln(2x-1)\,. \tag{30}$$

In the first integral on the right hand side of eq. (30), we substitute $y = 2x$, and in the second integral on the right hand side of eq. (30), we substitute $y = 2x - 1$. Hence,

$$F(z=4) = 2\int_0^1 \frac{dy}{y}\,\ln(1-y) + 2\int_0^1 \frac{dy}{1+y}\,\ln y\,. \tag{31}$$

The two integrals above are well known [9],

$$\int_0^1 \frac{dy}{y}\,\ln(1-y) = -\frac{\pi^2}{6}\,, \qquad \int_0^1 \frac{dy}{1+y}\,\ln y = -\frac{\pi^2}{12}\,. \tag{32}$$

Hence, it follows that

$$F(z=4) = -\tfrac{1}{2}\pi^2\,. \tag{33}$$

In light of eq. (26), $\lim_{\theta\to 0} F(\theta) = \lim_{z\to 4} F(z) = -\frac{1}{2}\pi^2$. Hence, it follows that we can extend the results of eqs. (28) and (29) to include the endpoint $z = 4$.

[1] An alternative derivation of eq. (28) and its analytic continuation are given in Appendix A.

Finally, we consider the case of $z>4$. In this case, $\operatorname{Im} F(z)\neq 0$ and is given explicitly in eq. (6). In order to compute $\operatorname{Re} F(z)$ when $z>4$, it is convenient to define,

$$z=4\cosh^2 w\,,\qquad \text{for } 0<w<\infty\,. \tag{34}$$

In light of eq. (5),

$$\operatorname{Re} F(z)=\int_0^1 \frac{dx}{x}\,\ln\bigl|1-zx(1-x)\bigr|\,. \tag{35}$$

After employing eq. (34), we take the derivative of $\operatorname{Re} F$ with respect to w,

$$\frac{d}{dw}\operatorname{Re} F=\frac{d}{dw}\int_0^1\frac{dx}{x}\,\ln\bigl|1-4x(1-x)\cosh^2 w\bigr|=-4\sinh 2w\,\mathrm{P}\int_0^1\frac{(1-x)dx}{1-4x(1-x)\cosh^2 w}\,, \tag{36}$$

where P indicates the principal value prescription. In obtaining this result, we have made use of the relation obtained on p. 26 of Ref. [10],

$$\frac{d}{dy}\ln|y|=\mathrm{P}\,\frac{1}{y}\,. \tag{37}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1-4x(1-x)\cosh^2 w=4\cosh^2 w(x-x_+)(x-x_-)\,,\qquad \text{where } x_\pm\equiv\frac{e^{\pm w}}{2\cosh w}. \tag{38}$$

Hence, it follows that

$$\frac{d}{dw}\operatorname{Re} F=-\frac{\sinh 2w}{\cosh^2 w}\,\mathrm{P}\int_0^1\frac{(1-x)dx}{(x-x_+)(x-x_-)}=-\frac{2\tanh w}{x_+-x_-}\,\mathrm{P}\int_0^1\left(\frac{1-x_+}{x-x_+}-\frac{1-x_-}{x-x_-}\right)dx\,. \tag{39}$$

Using eq. (38), it follows that

$$x_+-x_-=\tanh w\,,\qquad x_++x_-=1\,. \tag{40}$$

Moreover,

$$\mathrm{P}\int_0^1\frac{dx}{x-x_-}=\mathrm{P}\int_0^1\frac{dx}{x-1+x_+}=-\mathrm{P}\int_0^1\frac{dx}{x-x_+}\,, \tag{41}$$

after changing the integration variable $x\to 1-x$ in the final step above. Using the definition of the principal value prescription,

$$\begin{aligned}\mathrm{P}\int_0^1\frac{dx}{x-x_+}&=\lim_{\delta\to 0^+}\left\{\int_0^{x_+-\delta}\frac{dx}{x-x_+}+\int_{x_++\delta}^1\frac{dx}{x-x_+}\right\}\\ &=\lim_{\delta\to 0^+}\left\{\ln(x_+-x)\Big|_0^{x_+-\delta}+\ln(x-x_+)\Big|_{x_++\delta}^1\right\}\\ &=\lim_{\delta\to 0^+}\bigl\{\ln\delta-\ln x_++\ln(1-x_+)-\ln\delta\bigr\}\\ &=\ln\left(\frac{1-x_+}{x_+}\right)=\ln\left(\frac{x_-}{x_+}\right)=-2w\,,\end{aligned} \tag{42}$$

after making use of eqs. (38) and (40).

Finally, after employing eqs. (40)–(42), one can simplify eq. (39) to obtain,

$$\frac{d}{dw}\,\mathrm{Re}\,F = -2(2 - x_+ - x_-)\,\mathrm{P}\int_0^1 \frac{dx}{x - x_+} = 4w\,. \tag{43}$$

Integrating both sides of eq. (43) and using eq. (33) to determine the constant of integration,

$$\mathrm{Re}\,F(w) = 2w^2 - \tfrac{1}{2}\pi^2\,. \tag{44}$$

From eq. (34), $\cosh w = \frac{1}{2}\sqrt{z}$. Hence, it follows that

$$\mathrm{Re}\,F(z) = 2\left[\mathrm{arccosh}\left(\tfrac{1}{2}\sqrt{z}\right)\right]^2 - \tfrac{1}{2}\pi^2\,, \qquad \text{for real } z \geq 4\,, \tag{45}$$

where the principal value of the arccosh function for real positive values of $z \geq 4$ is

$$\mathrm{arccosh}\left(\tfrac{1}{2}\sqrt{z}\right) = \ln\left(\frac{\sqrt{z}}{2} + \sqrt{\frac{z}{4} - 1}\right)\,. \tag{46}$$

Note that eqs. (6) and (46) imply that

$$\mathrm{Im}\,F(z) = -2\pi\,\mathrm{arccosh}\left(\tfrac{1}{2}\sqrt{z}\right)\,, \qquad \text{for real } z \geq 4\,. \tag{47}$$

Combining eqs. (45) and (47) yields,

$$F(z) = -2\left[\frac{\pi}{2} + i\,\mathrm{arccosh}\left(\tfrac{1}{2}\sqrt{z}\right)\right]^2\,, \qquad \text{for } z > 4\,. \tag{48}$$

In summary,

$$F(z) = \begin{cases} -2\left[\arcsin\left(\tfrac{1}{2}\sqrt{z}\,\right)\right]^2\,, & \text{for } 0 \leq z \leq 4\,, \\ -2\left[\tfrac{\pi}{2} + i\,\mathrm{arccosh}\left(\tfrac{1}{2}\sqrt{z}\right)\right]^2\,, & \text{for } z > 4\,, \end{cases} \tag{49}$$

where $0 \leq \arcsin\left(\frac{1}{2}\sqrt{z}\right) \leq \frac{1}{2}\pi$. In the literature, one often rewrites the expression for $F(z)$ when $z > 4$ in one of the two following equivalent forms,[2]

$$F(z) = -2\left[\frac{\pi}{2} + i\ln\left(\frac{\sqrt{z}}{2} + \sqrt{\frac{z}{4} - 1}\right)\right]^2 = -\frac{1}{2}\left[\pi + i\ln\left(\frac{1 + \sqrt{1 - \frac{4}{z}}}{1 - \sqrt{1 - \frac{4}{z}}}\right)\right]^2\,, \qquad \text{for } z > 4, \tag{50}$$

after employing the identity,

$$\mathrm{arccosh}\,x = \ln\left(x + \sqrt{x^2 - 1}\right)\,, \qquad \text{for } x \geq 1. \tag{51}$$

It is straightforward to show that the two expressions on the right hand side of eq. (49) are analytic continuations of one another. This statement is proven at the end of Appendix A.

A similar method to the one presented in these notes for evaluating $F(z)$ has been given in Refs. [11,12]. In these two references, dF/dz is evaluated first and then the result is integrated to obtain $F(z)$. However, the final integration of dF/dz is more difficult as compared to the derivation given above.

Yet another technique for deriving $F(z)$ is given in Appendix C.

[2] This result corrects a sign error that appears in eq. (3.3) of Ref. [5], which gave the opposite sign for the imaginary part of $F(z)$. The sign was corrected in Ref. [6] (and also appears correctly on p. 434 of Ref. [7]).

Appendix A: Alternative derivation of $F(z)$

For $0 \leq z < 4$, $\operatorname{Im} F(z) = 0$. One can therefore drop the factor of $-i\epsilon$ in eq. (1) and write,

$$F(z) \equiv \int_0^1 \frac{dx}{x} \ln\bigl[1 - zx(1-x)\bigr]\,, \qquad \text{for } 0 \leq z < 4. \tag{52}$$

Noting that $0 \leq zx(1-x) < 1$ for all $0 \leq x \leq 1$ and $0 \leq z < 4$, we can employ a series expansion for the logarithm,

$$\ln(1-w) = -\sum_{n=1}^{\infty} \frac{w^n}{n}\,, \qquad \text{for } -1 \leq w < 1. \tag{53}$$

Setting $w = zx(1-x)$ in eq. (52) and interchanging the order of integration and summation,

$$F(z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^1 x^{n-1}(1-x)^n\, dx\,, \qquad \text{for } 0 \leq z < 4. \tag{54}$$

We recognize the integral above as a beta function,

$$B(n, n+1) = \int_0^1 x^{n-1}(1-x)^n\, dx = \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} = \frac{(n-1)!\, n!}{(2n)!}\,. \tag{55}$$

Plugging this result back into eq. (54) yields,

$$F(z) = -\sum_{n=1}^{\infty} \frac{\bigl[(n-1)!\bigr]^2}{(2n)!}\, z^n\,, \qquad \text{for } 0 \leq z < 4. \tag{56}$$

Comparing this result with eq. (71) of Appendix B, we conclude that

$$F(z) = -2\bigl[\arcsin\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\bigr]^2\,, \qquad \text{for } 0 \leq z < 4\,, \tag{57}$$

where arcsin is the principal value of the arcsine function, which satisfies $|\arcsin x| \leq \frac{1}{2}\pi$ for real values of x. Thus, we have confirmed the result of eq. (28).

One can now employ the method of analytic continuation to obtain $F(z)$ in the region where $z > 4$. Note that an equivalent form for eq. (57) is,

$$F(z) = -2\left[\frac{\pi}{2} - \arccos\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\right]^2\,, \qquad \text{for } 0 \leq z < 4\,. \tag{58}$$

To analytically continue into the region of real $z > 4$, we employ eqs. (4.23.24) and (4.37.19) of Ref. [13], which imply that for a positive infinitesimal ϵ,

$$\lim_{\epsilon \to 0^+} \arccos(x + i\epsilon) = -i \operatorname{arccosh} x = -i \ln\bigl(x + \sqrt{x^2-1}\bigr)\,, \qquad \text{for } 1 < x < \infty. \tag{59}$$

Consequently, for $z > 4$,

$$F(z) = \lim_{\epsilon \to 0^+} F(z + i\epsilon) = -2\left[\frac{\pi}{2} - \arccos\bigl(\tfrac{1}{2}\sqrt{z} + i\epsilon\bigr)\right]^2 = -2\left[\frac{\pi}{2} + i \operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\right]^2, \tag{60}$$

in agreement with eq. (48). As expected, both eqs. (58) and (60) yield the same result at their common boundary, $F(z=4) = -\frac{1}{2}\pi^2$.

Similarly, the analytic continuation of eq. (57) into the region of $z < 0$ yields eq. (14) in light of the relation, $\arcsin(ix) = i \operatorname{arcsinh} x$.

A careful treatment of the analytic continuation is also given in Ref. [12].

Appendix B: Power series of $(\arcsin x)^2$

One method for deriving a power series of a function is to develop a differential equation (with appropriate initial conditions) whose solution is the function in question. This differential equation can then be solved by the series expansion method. This technique was used by Ref. [14] to derive the Taylor series for $(\arcsin x)^2$ about the origin.[3] Inspired by the computation of Ref. [14], we first consider the function,

$$y = \frac{\arcsin x}{\sqrt{1-x^2}}\,, \tag{61}$$

where the principal value of the arcsine function is employed such that $|\arcsin x| \leq \frac{1}{2}\pi$ for real values of x. We can derive the Taylor series of eq. (61) about $x = 0$ by the following technique. Taking the derivative of eq. (61) yields

$$\frac{dy}{dx} = \frac{1}{1-x^2} + \frac{x\,\arcsin x}{(1-x^2)^{3/2}}\,. \tag{62}$$

It follows that eq. (61) is the solution to the following first order differential equation,

$$(1-x^2)\frac{dy}{dx} - xy = 1\,, \qquad \text{where } y(x=0) = 0\,. \tag{63}$$

Note that setting $x = 0$ in eq. (61) yields $y = 0$ which fixes the initial condition for eq. (63).

One can solve eq. (63) using a series solution,

$$y = \sum_{n=0}^{\infty} c_n x^n\,. \tag{64}$$

Plugging eq. (64) back into eq. (63) yields,

$$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=1}^{\infty} nc_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 1\,. \tag{65}$$

Equating coefficients of x^n on both sides of eq. (65) and imposing $y(x=0) = 0$ yields $c_0 = 0$, $c_1 = 1$ and

$$c_{n+1} = \frac{n}{n+1}c_{n-1}\,, \quad \text{for } n = 1, 2, 3, \ldots \tag{66}$$

It immediately follows that

$$c_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3} = \frac{2^n n!}{(2n+1)!!}\,, \qquad c_{2n} = 0\,, \qquad \text{for } n = 0, 1, 2, \ldots \tag{67}$$

Hence we conclude that

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!!}\,x^{2n+1}\,, \qquad \text{for } |x| < 1. \tag{68}$$

In eq. (68), we have noted that the convergence of the sum requires that $|x| < 1$.

[3] Other methods for obtaining the Taylor series for $(\arcsin x)^2$ about $x = 0$ can be found in Refs. [15–19].

In light of

$$\frac{d}{dx}\,(\arcsin x)^2 = \frac{2\arcsin x}{\sqrt{1-x^2}}\,,$$

it follows that

$$(\arcsin x)^2 = 2\int_0^x \frac{\arcsin t}{\sqrt{1-t^2}}\,dt\,. \tag{69}$$

Inserting the series obtained in eq. (68) on the right hand side of eq. (69) yields,

$$\begin{aligned}(\arcsin x)^2 &= 2\sum_{n=0}^\infty \frac{2^n n!}{(2n+1)!!}\int_0^x t^{2n+1} = \sum_{n=0}^\infty \frac{2^n n!}{(n+1)(2n+1)!!}\,x^{2n+2} \\ &= \sum_{n=1}^\infty \frac{2^{n-1}(n-1)!}{n(2n-1)!!}\,x^{2n}\,, \qquad \text{for } |x|\leq 1\,. \end{aligned} \tag{70}$$

One can check that the series on the right hand side of eq. (70) converges at all points on the boundary of the circle of convergence.

Note that

$$(2n)! = (2n)!!\,(2n-1)!! = 2^n n!\,(2n-1)!!\,.$$

Hence,

$$(2n-1)!! = \frac{(2n)!}{2^n n!}\,.$$

Inserting this result into eq. (70) yields,

$$(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^\infty \frac{\big[(n-1)!\big]^2}{(2n)!}\,(2x)^{2n}\,, \qquad \text{for } |x|\leq 1\,. \tag{71}$$

Using $(n-1)! = n!/n$ and introducing the central binomial coefficient,

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}\,,$$

one can rewrite eq. (71) as

$$(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^\infty \frac{(2x)^{2n}}{n^2\binom{2n}{n}}\,, \qquad \text{for } |x|\leq 1\,. \tag{72}$$

Appendix C: Yet another derivation of $F(z)$

In this Appendix, we shall evaluate $F(z)$ defined in eq. (1) by employing the dilogarithm function discussed in detail in Ref. [20]. Eq. (1) is a special case of a more general integral with an integrand that consists of a ratio of quadratic polynomials. Such an integral was examined previously in Ref. [21]. In Appendix D, we present a detailed computation of this more general integral and correct some typographical errors that appear in Ref. [21].

Using eqs. (2) and (3),

$$f(x) = \frac{(x_+ - x)(x_- - x)}{x_+ x_-} = \left(1 - \frac{x}{x_+}\right)\left(1 - \frac{x}{x_-}\right) = (1 - xzx_+)\,(1 - xzx_-)\;, \tag{73}$$

after making use of $x_+ x_- = 1/z$.

In the region where $0 \leq z < 4$,

$$x_\pm = \tfrac{1}{2}\left[1 \pm i\sqrt{\frac{4}{z} - 1}\,\right]. \tag{74}$$

Moreover, $f(x)$ is strictly positive for $0 \leq x \leq 1$, in which case we can safely take the limit of $\epsilon \to 0$ in eq. (1). Hence,

$$F(z) = \int_0^1 \frac{dx}{x}\,\ln\bigl[f(x)\bigr] = \int_0^1 \frac{dx}{x}\,\ln(1 - xzx_+) + \int_0^1 \frac{dx}{x}\,\ln(1 - xzx_-)\;. \tag{75}$$

Using the following indefinite integral taken from Ref. [20],

$$\int^x \frac{dx}{x}\,\ln(1 - Ax) = -\operatorname{Li}_2(Ax)\,, \tag{76}$$

where $\operatorname{Li}_2(x)$ is the dilogarithm function, it follows that

$$F(z) = -\operatorname{Li}_2(zx_+) - \operatorname{Li}_2(zx_-)\,. \tag{77}$$

Note that

$$zx_\pm = \tfrac{1}{2}\left[z \pm i\sqrt{4z - z^2}\right] = \sqrt{z}\,e^{\pm i\phi}\,, \qquad \text{where } \tan\phi = \sqrt{\frac{4}{z} - 1}\,. \tag{78}$$

Following Ref. [20], we define

$$\operatorname{Li}_2(r,\phi) \equiv \operatorname{Re}\operatorname{Li}(re^{i\phi}) = -\frac{1}{2}\int_0^r \frac{\ln(1 - 2x\cos\phi + x^2)}{x}\,dx\,. \tag{79}$$

Since $F(z)$ is real in the region where $0 < z < 4$, eq. (77) yields,

$$F(z) = \operatorname{Re}F(z) = -\operatorname{Re}\bigl[\operatorname{Li}_2(zx_+) + \operatorname{Li}_2(zx_-)\bigr] = -\bigl[\operatorname{Li}_2(\sqrt{z},\phi) + \operatorname{Li}_2(\sqrt{z},-\phi)\bigr]\,. \tag{80}$$

Eq. (79) implies that

$$\operatorname{Li}_2(r,\phi) = \operatorname{Li}_2(r,-\phi)\,. \tag{81}$$

Hence,

$$F(z) = -2\,\operatorname{Li}_2(\sqrt{z},\phi)\,, \qquad \text{where } \tan\phi = \sqrt{\frac{4}{z} - 1} \text{ and } 0 \leq z < 4\,. \tag{82}$$

Since $0 \leq z < 4$, it follows from eq. (78) that $\operatorname{Re}(zx_\pm) \geq 0$. Hence, $\cos\phi \geq 0$, in which case

$$\cos\phi = \frac{1}{\sqrt{1 + \tan^2\phi}} = \tfrac{1}{2}\sqrt{z}\,. \tag{83}$$

Eqs. (82) and (83) then yield,

$$F(z) = -2\ \mathrm{Li}_2(2\cos\phi, \phi)\,, \qquad \text{where } \cos\phi = \tfrac{1}{2}\sqrt{z} \text{ and } 0 \leq z < 4\,. \tag{84}$$

Finally, we can employ formula (15) of Ref. [20], which states that

$$\mathrm{Li}_2(2\cos\phi, \phi) = \left(\tfrac{1}{2}\pi - \phi\right)^2, \qquad \text{for } 0 \leq \phi \leq \pi. \tag{85}$$

In light of the fact that

$$\tfrac{1}{2}\pi - \phi = \tfrac{1}{2}\pi - \arccos\left(\tfrac{1}{2}\sqrt{z}\right) = \arcsin\left(\tfrac{1}{2}\sqrt{z}\right), \tag{86}$$

it follows from eqs. (84)–(86) that

$$F(z) = -2\left[\arcsin\left(\tfrac{1}{2}\sqrt{z}\right)\right]^2, \qquad \text{for } 0 \leq z < 4\,, \tag{87}$$

in agreement with eq. (28).

In the region where $z > 4$, one cannot drop the factor of $-i\epsilon$ in eq. (1). Noting that both roots,

$$x_\pm = \tfrac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{z}}\,\right], \tag{88}$$

are real and lie between 0 and 1, we see that eq. (75) must be modified as follows,

$$F(z) = \int_0^1 \frac{dx}{x}\,\ln\left(1 - xzx_+ - i\epsilon\right) + \int_0^1 \frac{dx}{x}\,\ln\left(1 - xzx_- + i\epsilon\right)\,. \tag{89}$$

In particular, note that since $0 < x_- < x_+ < 1$ and $z > 4$, it follows that

$$(1 - xzx_+ - i\epsilon)(1 - xzx_- + i\epsilon) = 1 - zx(1-x) + i\epsilon\, xz(x_- - x_+) = 1 - zx(1-x) - i\epsilon\,, \tag{90}$$

in a region where $0 < x < 1$, as required. An alternative way to arrive at the same result is to note that $z \to z + i\epsilon$ is equivalent to $x_\pm \to x_\pm \pm i\epsilon$. Since $zx_\pm = 1/x_\mp$, eq. (76) yields,

$$F(z) = -\,\mathrm{Li}_2(zx_+ + i\epsilon) - \mathrm{Li}_2(zx_- - i\epsilon)\,, \qquad \text{for } z > 4. \tag{91}$$

Using $zx_+x_- = 1$ and $x_+ + x_- = 1$, it follows that

$$zx_- - i\epsilon = \frac{1}{x_+} - i\epsilon = \frac{1}{1 - x_-} - i\epsilon = \frac{1}{1 - (zx_+)^{-1}} = \frac{zx_+}{zx_+ - 1} - i\epsilon\,. \tag{92}$$

Thus, we can rewrite eq. (91) as,

$$F(z) = -\,\mathrm{Li}_2(zx_+ + i\epsilon) - \mathrm{Li}_2\left(\frac{zx_+}{zx_+ - 1} - i\epsilon\right), \qquad \text{for } z > 4. \tag{93}$$

The factors involving $i\epsilon$ indicate how to evaluate the dilogarithms which are defined on the complex plane with a branch cut from 1 to ∞. Specifically, for real values of x and positive infinitesimal ϵ,

$$\mathrm{Li}_2(x \pm i\epsilon) = \mathrm{Re}\,\mathrm{Li}_2(x) \pm i\pi\,\Theta(x-1)\ln x\,. \tag{94}$$

The step function $\Theta(x-1)$ in eq. (94) indicates that the dilogarithm of a real number possesses an imaginary part only when the argument of the dilogarithm is greater than 1.

Formula (9) on p. 283 of Ref. [20], with the imaginary part modified appropriately[4] in light of eq. (94), states that

$$\mathrm{Re}\left[\mathrm{Li}_2(x)+\mathrm{Li}_2\left(\frac{x}{x-1}\right)\right]=\tfrac{1}{2}\pi^2-\tfrac{1}{2}\ln^2(x-1)\,,\qquad \text{for } x>1, \tag{95}$$

and

$$\mathrm{Im}\left[\mathrm{Li}_2(x+i\epsilon)+\mathrm{Li}_2\left(\frac{x}{x-1}-i\epsilon\right)\right]=\pi\ln(x-1)\,,\qquad \text{for } x>1. \tag{96}$$

Hence, eq. (93) yields,

$$F(z)=-\tfrac{1}{2}\pi^2+\tfrac{1}{2}\ln^2(zx_+-1)-i\pi\ln(zx_+-1)=-\tfrac{1}{2}\big[\pi+i\ln^2(zx_+-1)\big]^2. \tag{97}$$

Noting that

$$zx_+-1=\frac{1}{x_-}-1=\frac{1-x_-}{x_-}=\frac{x_+}{x_-}\,, \tag{98}$$

it follows that

$$F(z)=-\frac{1}{2}\left[\pi+i\ln\left(\frac{x_+}{x_-}\right)\right]^2,\qquad \text{for } z>4. \tag{99}$$

Employing eq. (88), we end up with

$$F(z)=-\frac{1}{2}\left[\pi+i\ln\left(\frac{1+\sqrt{1-\frac{4}{z}}}{1-\sqrt{1-\frac{4}{z}}}\right)\right]^2,\qquad \text{for } z>4, \tag{100}$$

in agreement with eq. (50).

Finally, we can also treat the region of $z<0$. Using eqs. (77) and (92),

$$F(z)=-\mathrm{Li}_2(zx_+)-\mathrm{Li}_2\left(\frac{zx_+}{zx_+-1}\right),\qquad \text{for } z<0. \tag{101}$$

Note that $zx_+\in(0,1)$ when $z<0$, so the arguments of both dilogarithms are less than 1, which means that $F(z)$ is real, as expected from eq. (6). Thus, we can evaluate eq. (101) by employing formula (8) on p. 283 of Ref. [20], which states that

$$\mathrm{Li}_2(x)+\mathrm{Li}_2\left(-\frac{x}{1-x}\right)=-\tfrac{1}{2}\ln^2(1-x)\,,\qquad \text{for } x<1. \tag{102}$$

Hence, eq. (101) yields,

$$F(z)=\tfrac{1}{2}\ln^2(1-zx_+)\,,\qquad \text{for } x<1. \tag{103}$$

[4] In formula (9) on p. 283 of Ref. [20], it has been implicitly assumed that x means $x-i\epsilon$ in *both* the arguments of the dilogarithms.

Using eq. (98), it follows that

$$1 - zx_+ = -\frac{x_+}{x_-} = \frac{\sqrt{1-\frac{4}{z}}+1}{\sqrt{1-\frac{4}{z}}-1}. \tag{104}$$

Thus, we end up with,

$$F(z) = \frac{1}{2}\ln^2\left(\frac{\sqrt{1-\frac{4}{z}}+1}{\sqrt{1-\frac{4}{z}}-1}\right), \qquad \text{for } z<0, \tag{105}$$

in agreement with eq. (16).

Appendix D: Generalizing the results of Appendix C

The integral treated in these notes is a special case of the more general integral,

$$G(x_1;b,c) = \int_0^1 \frac{dx}{x-x_1}\ln\left(\frac{x^2+bx+c-i\epsilon}{x_1^2+bx_1+c-i\epsilon}\right), \tag{106}$$

where x_1, b and c are real parameters and ϵ is an infinitesimal constant (which can be either positive or negative).[5] Note that the potential singularity of the integrand at $x = x_1$ is removable. One can evaluate $G(x_1;a,b,c)$ by using the same techniques employed in Appendix C. In this Appendix, we will provide details of the evaluation of eq. (106), and verify that it reduces to the results obtained in Appendix C for $G(0;-1,1/z)$.

We first consider the case of $b^2 < 4c$. In this case, $c > 0$ and the polynomial equation $x^2+bx+c=0$ has two complex roots,

$$x_\pm = \tfrac{1}{2}\left[-b \pm i\sqrt{4c-b^2}\right], \tag{107}$$

which implies that $x^2+bx+c>0$ for all x. Consequently, it is safe to take the limit of $\epsilon \to 0$. After factoring the argument of the logarithm,

$$G(x_1;b,c) = G(x_1,x_+) + G(x_1,x_-), \tag{108}$$

where

$$G(x_1,x_\pm) = \int_0^1 \frac{dx}{x-x_1}\ln\left(\frac{x-x_\pm}{x_1-x_\pm}\right), \tag{109}$$

which consists of two equations (one for the upper signs and one for the lower signs). Changing the integration variable to $y = x - x_1$,

$$G(x_1,x_\pm) = \int_{-x_1}^{1-x_1} \frac{dy}{y}\ln\left(1-\frac{y}{x_\pm - x_1}\right). \tag{110}$$

[5] In the more general case where the argument of the logarithm is $(ax^2+bx+c-i\epsilon)/(ax_1^2+bx_1+c-i\epsilon)$ where $a \neq 0$, one can always divide the numerator and denominator by a and change the sign of ϵ if $a<0$. Thus, without loss of generality, one is free to consider the form of the integral as given by eq. (106).

Employing eq. (76), we end up with

$$G(x_1, x_\pm) = -\operatorname{Li}_2\left(\frac{1-x_1}{x_\pm - x_1}\right) + \operatorname{Li}_2\left(\frac{-x_1}{x_\pm - x_1}\right) . \tag{111}$$

Note that

$$\frac{1-x_1}{x_\pm - x_1} = \frac{|1-x_1|}{\sqrt{x_1 + bx_1 + c}}\, e^{\pm i\phi} , \qquad \text{for } \phi = \theta + \pi\,\Theta(x_1 - 1) , \tag{112}$$

$$\frac{-x_1}{x_\pm - x_1} = \frac{|x_1|}{\sqrt{x_1 + bx_1 + c}}\, e^{\pm i\phi'} , \qquad \text{for } \phi' = \theta + \pi\,\Theta(x_1) , \tag{113}$$

where

$$\cos\theta = -\frac{2x_1 + b}{2\sqrt{x_1^2 + bx_1 + c}} . \tag{114}$$

To derive the above results, we calculate as follows,

$$\frac{1-x_1}{x_+ - x_1} = \frac{(1-x_1)(x_- - x_1)}{(x_+ - x_1)(x_- - x_1)} = \frac{1-x_1}{2(x_1^2 + bx_1 + c)}\left[-2x_1 - b - i\sqrt{4c - b^2}\right] . \tag{115}$$

In a convention where $-\pi < \phi \leq \pi$, it follows that

$$\tan\phi = \frac{\sqrt{4c - b^2}}{2x_1 + b} , \tag{116}$$

and the two-fold ambiguity on the determination of ϕ is fixed by the sign of $\cos\phi$. In particular, the sign of $\cos\phi$ is equal to the sign of $\operatorname{Re}\big[(1-x_1)/(x_+ - x_1)\big]$. That is,

$$\operatorname{sgn}(\cos\phi) = \begin{cases} +1 , & \text{if } (1-x_1)(2x_1 + b) < 0 , \\ -1 , & \text{if } (1-x_1)(2x_1 + b) > 0 . \end{cases} \tag{117}$$

It follows that,

$$\begin{aligned} \cos\phi &= -\frac{1}{\sqrt{1 + \tan^2\phi}}\, \operatorname{sgn}\big((1-x_1)(2x_1 + b)\big) = -\frac{2x_1 + b}{2\sqrt{x_1^2 + bx_1 + c}}\operatorname{sgn}(1 - x_1) \\ &= \cos\theta\, \operatorname{sgn}(1 - x_1) = \cos\big(\theta + \pi\,\Theta(x_1 - 1)\big) , \end{aligned} \tag{118}$$

in agreement with the result for $(1-x_1)/(x_+ - x_1)$ given in eq. (112). The other cases specified in eqs. (112) and (113) can be obtained similarly.

Hence, after using the definition of $\operatorname{Li}_2(r, \phi)$ given in eq. (79) and making use of eq. (81), we end up with

$$G(x_1; b, c) = 2\operatorname{Li}_2\left(\frac{|x_1|}{\sqrt{x_1 + bx_1 + c}}, \theta + \pi\,\Theta(x_1)\right) - 2\operatorname{Li}_2\left(\frac{|1-x_1|}{\sqrt{x_1 + bx_1 + c}}, \theta + \pi\,\Theta(x_1 - 1)\right) . \tag{119}$$

It is convenient to allow the parameter x in $\operatorname{Li}_2(x, \phi)$ to be negative. Then, eq. (79) implies that

$$\operatorname{Li}_2(-x, \theta) = \operatorname{Li}_2(x, \theta + \pi) . \tag{120}$$

In addition, note that

$$\mathrm{Li}_2(x,0) = \mathrm{Re}\,\mathrm{Li}_2(x)\,, \tag{121}$$

which means that $\mathrm{Li}_2(x,0) \neq \mathrm{Li}_2(x)$ for $x > 1$, since in this regime $\mathrm{Li}_2(x)$ possesses an imaginary part, whereas $\mathrm{Li}_2(x,0)$ is real.

It then follows that,

$$G(x_1;b,c) = 2\,\mathrm{Li}_2\left(\frac{-x_1}{\sqrt{x_1+bx_1+c}},\theta\right) - 2\,\mathrm{Li}_2\left(\frac{1-x_1}{\sqrt{x_1+bx_1+c}},\theta\right)\,, \tag{122}$$

where

$$\cos\theta = -\frac{2x_1+b}{2\sqrt{x_1^2+bx_1+c}}\,, \tag{123}$$

in agreement with the result given in Appendix D of Ref. [21].

As a check, we set $x_1 = 0$, $b = -1$ and $c = 1/z$, under the assumption that $0 \leq z < 4$, to obtain,

$$G(0;-1,1/z) = -2\,\mathrm{Li}_2(\sqrt{z},\theta)\,, \qquad \text{where } \cos\theta = \tfrac{1}{2}\sqrt{z} \text{ and } 0 \leq z < 4, \tag{124}$$

in agreement with the result of eq. (84).

Next, we consider the case of $b^2 > 4c$. In this case, the polynomial equation $x^2+bx+c = 0$ has two real roots,

$$x_\pm = \tfrac{1}{2}\left[-b \pm \sqrt{b^2-4c}\right]. \tag{125}$$

Depending on the values of x_+, x_- and x_1, the argument of the logarithm may change sign in the interval $0 \leq x \leq 1$. If this happens, then the logarithm will develop an imaginary part, whose sign is fixed by the presence of the $i\epsilon$ term. However, it will be convenient to first consider the case where the argument of the logarithm is strictly positive in the interval of $0 \leq x \leq 1$. In this case we can safely take the limit of $\epsilon \to 0$. Once we obtain the result for this case, one can analytically continue the result into other parameter regimes simply by replacing $c \to c - i\epsilon$.

Thus, if we assume that $x_\pm < x_1$ and $x_\pm < 0$, then we can factor the argument of the logarithm as we did in eqs. (108)–(110), and we once again arrive at eq. (111),

$$G(x_1, x_\pm) = -\,\mathrm{Li}_2\left(\frac{1-x_1}{x_\pm - x_1}\right) + \mathrm{Li}_2\left(\frac{-x_1}{x_\pm - x_1}\right). \tag{126}$$

As expected, both dilogarithm arguments are less than 1, which confirms that the functions $G(x_1, x_\pm)$ are real, as expected. Hence eq. (108) yields,

$$G(x_1;b,c) = \mathrm{Li}_2\left(\frac{-x_1}{x_+-x_1}\right) + \mathrm{Li}_2\left(\frac{-x_1}{x_--x_1}\right) - \mathrm{Li}_2\left(\frac{1-x_1}{x_+-x_1}\right) - \mathrm{Li}_2\left(\frac{1-x_1}{x_--x_1}\right). \tag{127}$$

If we drop the assumption that $x_\pm < x_1$ and $x_\pm < 0$, then some of the dilogarithm arguments may be larger than 1 indicating that an imaginary part is present, in light of eq. (94). Thus, we now restore the $i\epsilon$ factors by letting $c \to c - i\epsilon$. In particular, we must replace $x_\pm \to x_\pm \pm i\epsilon$. For example,

$$\frac{-x_1}{x_\pm - x_1 \pm i\epsilon} = \frac{-x_1}{(x_\pm - x_1)\left(1 \pm \dfrac{i\epsilon}{x_\pm - x_1}\right)} = \frac{-x_1}{x_\pm - x_1} \pm \frac{i\epsilon\, x_1}{(x_\pm - x_1)^2} = \frac{-x_1}{x_\pm - x_1} \pm i\epsilon \operatorname{sgn} x_1 \,. \tag{128}$$

Hence, it follows that independently of the values of $x_\pm$ and x_1,[6]

$$\begin{aligned} G(x_1; b, c) = {} & \mathrm{Li}_2\left(\frac{-x_1}{x_+ - x_1} + i\epsilon \operatorname{sgn} x_1\right) + \mathrm{Li}_2\left(\frac{-x_1}{x_- - x_1} - i\epsilon \operatorname{sgn} x_1\right) \\ & - \mathrm{Li}_2\left(\frac{1 - x_1}{x_+ - x_1} + i\epsilon \operatorname{sgn}(x_1 - 1)\right) - \mathrm{Li}_2\left(\frac{1 - x_1}{x_- - x_1} - i\epsilon \operatorname{sgn}(x_1 - 1)\right). \end{aligned} \tag{129}$$

As a check, we take $x_1 = 0$, $b = -1$ and $c = 1/z$, which yields, $zx_+x_- = zc = 1$ and

$$G(0; -1, 1/z) = -\mathrm{Li}_2(zx_+ + i\epsilon) - \mathrm{Li}_2(zx_- - i\epsilon)\,, \qquad \text{for } z > 4, \tag{130}$$

in agreement with eq. (91).

For completeness, we examine a third case where $b^2 = 4c$. In this case, it is safe to take the $\epsilon \to 0$ limit, which yields,

$$\begin{aligned} G\big(x_1; b, \tfrac{1}{4}b^2\big) &= 2\int_0^1 \frac{dx}{x - x_1} \ln\left|\frac{x + \frac{1}{2}b}{x_1 + \frac{1}{2}b}\right| \\ &= 2\,\mathrm{Re}\left\{\int_{-x_1}^{1-x_1} \frac{dy}{y} \ln\left(1 + \frac{y}{x_1 + \frac{1}{2}b}\right)\right\} \\ &= 2\,\mathrm{Re}\left\{\mathrm{Li}_2\left(\frac{x_1}{x_1 + \frac{1}{2}b}\right) - \mathrm{Li}_2\left(\frac{x_1 - 1}{x_1 + \frac{1}{2}b}\right)\right\}\,, \end{aligned} \tag{131}$$

after making use of eqs. (24) and (76). It is a simple exercise to check that in the limit of $b^2 = 4c$, eqs. (122) and (129) both reduce to eq. (131), as expected.[7]

If the argument of the dilogarithm is larger than 1, then one can use formula (6) on p. 283 of Ref. [20] to write,

$$\mathrm{Re}\,\mathrm{Li}_2(x) = \tfrac{1}{3}\pi^2 - \mathrm{Li}_2\left(\frac{1}{x}\right) - \tfrac{1}{2}\ln^2 x\,, \qquad \text{for } x > 1. \tag{132}$$

The reader can rewrite the result of eq. (131) accordingly.

As one final check of the formulae derived in this Appendix, consider the limit where $|b|$, $|c| \to \infty$ with $r \equiv c/b$ held fixed to a finite value. In this limit, one can neglect the factor of x^2 in the numerator and denominator of eq. (106), in which case,

$$G(x_1; b, c) = \int_0^1 \frac{dx}{x - x_1} \ln\left(\frac{x + r - i\epsilon \operatorname{sgn} b}{x_1 + r - i\epsilon \operatorname{sgn} b}\right). \tag{133}$$

[6]Eq. (129) corrects a typographical error that appears in eq. (D.6) of Appendix D in Ref. [21].

[7]When analyzing eq. (122), we have made use of eqs. (120) and (121). When analyzing eq. (129), we have used the fact that $\mathrm{Li}_2(x + i\epsilon) + \mathrm{Li}_2(x - i\epsilon) = 2\,\mathrm{Re}\,\mathrm{Li}_2(x)$, in light of eq. (94).

Suppose that x_1 and r take on values such that $(x+r)/(x_1+r) > 0$ over the interval $0 < x < 1$. In this case, we can safely take the limit of $\epsilon \to 0$. Hence,

$$G(x_1; b, c) = \int_0^1 \frac{dx}{x - x_1} \ln\left(\frac{x+r}{x_1+r}\right) = \int_{-x_1}^{1-x_1} \frac{dy}{y} \ln\left(1 + \frac{y}{x_1+r}\right)$$
$$= \mathrm{Li}_2\left(\frac{x_1}{x_1+r}\right) - \mathrm{Li}_2\left(\frac{x_1-1}{x_1+r}\right) . \tag{134}$$

To obtain a result that is independent of the values of x_1 and r, one must restore the $i\epsilon$ factor by letting $r \to r - i\epsilon \operatorname{sgn} b$. Hence, we end up with,

$$G(x_1; b, c) = \mathrm{Li}_2\left(\frac{x_1}{x_1+r} + i\epsilon \operatorname{sgn}(bx_1)\right) - \mathrm{Li}_2\left(\frac{x_1-1}{x_1+r} + i\epsilon \operatorname{sgn}\big(b(x_1-1)\big)\right) . \tag{135}$$

This result corrects a typographical error in eq. (D.9) of Appendix D of Ref. [21] (the latter was valid only in the case of $b < 0$). One can verify that eq. (135) is reproduced when evaluating eq. (129) in the limit of $|b|, |c| \to \infty$ with $r \equiv c/b$. In performing this check, note that if $b > 0$ then $x_- \to \infty$ and $x_+ \to -r$, whereas if $b < 0$ then $x_+ \to \infty$ and $x_- \to -r$.

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