Notes on the spontaneous breaking of SU(N) and SO(N) via a second-rank tensor multiplet

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Abstract

The group theory of the spontaneous breaking of SU(N) is explored. Two specific cases are analyzed in detail: (i) SU(N) is broken to SO(N) via a scalar field vacuum expectation value for a second-rank symmetric tensor multiplet, and (ii) SU($2N$) or SU($2N+1$) is broken to Sp($2N$) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet. The case of the spontaneous breaking of SO($2N$) or SO($2N+1$) to U(N) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is also treated.

1. Introduction

In these notes, I study the group theory of the spontaneous breaking of a global SU(N)-symmetric field theory via a scalar field vacuum expectation value for second-rank tensor multiplet, $\langle\Sigma\rangle \equiv \Sigma_0$. The cases of a symmetric tensor and anti-symmetric tensor field are separately examined. I focus on one particular symmetry breaking pattern in each case corresponding to the maximal degeneracy of non-zero eigenvalues of $\Sigma_0^\dagger\Sigma_0$. The case of spontaneous breaking of SO($2N$) or SO($2N+1$) to U(N) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is very similar to the corresponding breaking of SU($2N$) or SU($2N+1$). A previous analysis of these (and other) cases can be found in ref. [1].

2. Symmetry breaking via a second-rank symmetric tensor

Let Σ^{ab} be a symmetric second-rank tensor which transforms under SU(N) as:

$$\Sigma^{ab} \longrightarrow U^a{}_c\, U^b{}_d\, \Sigma^{bd}\,, \tag{1}$$

where U is an $N \times N$ unitary matrix with unit determinant. Equivalently, in matrix form $\Sigma \longrightarrow U\Sigma U^T$, where U^T is the transpose of U. Suppose that Σ is a multiplet of scalar fields whose Lagrangian is invariant under global SU(N) transformations. If Σ acquires a vacuum expectation value $\langle\Sigma\rangle \equiv \Sigma_0$, then the SU($N$) symmetry will be broken. If there exists a subgroup H of SU(N), such that $U\Sigma_0 U^T = U$ for all $U \in H$, then the global SU(N) symmetry is spontaneously broken to H. Writing $U = \exp i\theta_a T_a$ where the T_a are the unbroken generators (which span the unbroken subgroup H), it follows that for infinitesimal θ_a,

$$(1 + i\theta_a T_a)\Sigma_0(1 + i\theta_a T_a^T) = \Sigma_0\,, \tag{2}$$

which implies that

$$T_a\Sigma_0 + \Sigma_0 T_a^T = 0\,. \tag{3}$$

Thus, it is possible to find a basis for the traceless hermitian SU(N) generators given by $\{T_a, X_b\}$ such that the T_a satisfy eq. (3). In this basis, the broken generators X_b are orthogonal to the T_a, that is Tr $(T_a X_b) = 0$.

The identity of the unbroken subgroup H depends on the choice of Σ_0 (which depends on the underlying dynamics responsible for the spontaneous symmetry breaking). Here, I shall consider the case of H=SO(N), for which the most general form for Σ_0 is a complex symmetric $N \times N$ matrix that satisfies $\Sigma_0^\dagger\Sigma_0 = \Sigma_0\Sigma_0^\dagger = |c|^2 I_N$, where $c \in \mathbb{C}$ and I_N is the $N \times N$ unit matrix. This is summarized by the following theorem.

Theorem: Suppose that Σ_0 is an $N \times N$ complex symmetric matrix that satisfies $\Sigma_0^\dagger\Sigma_0 = \Sigma_0\Sigma_0^\dagger = |c|^2 I_N$ for some complex number c. Then, if the generators of SU(N) in the defining (N-dimensional) representation are given by $\{T_a, X_b\}$, where the T_a and X_b are traceless hermitian $N \times N$ matrices that satisfy:

$$T_a\Sigma_0 + \Sigma_0 T_a^T = 0\,, \tag{4}$$

$$X_b\Sigma_0 - \Sigma_0 X_b^T = 0\,, \tag{5}$$

then the T_a span an unbroken SO(N) Lie subalgebra, while the X_b are the broken generators that span an SU(N)/SO(N) homogeneous space. Furthermore, Tr $(T_a X_b) = 0$.

Proof: First, I show that if $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_N$ and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an SO(N) Lie algebra. Note that these two conditions imply:

$$|c|^2\, T_a^T = -\Sigma_0^\dagger T_a \Sigma_0\,. \tag{6}$$

At this point, I note that the Takagi factorization [2, 3] for any complex symmetric matrix M corresponds to the statement that there exists a unitary matrix V such that VMV^T is diagonal with non-negative entries given by the positive square roots of the eigenvalues of $MM^\dagger$ (or $M^\dagger M$). In our case, this result implies that there exists a unitary matrix V such that[1]

$$V\Sigma_0 V^T = c\, I_N\,. \tag{7}$$

The inverse of this result is $(V^T)^{-1}\Sigma_0^\dagger V^{-1} = c^*\, I_N$ (since $\Sigma_0^\dagger = |c|^2\Sigma_0^{-1}$). I now define: $\widetilde{T}_a \equiv V T_a V^{-1}$, where V is the unitary matrix appearing in eq. (7). Then, inserting this result into eq. (6), it follows that:

$$\begin{aligned}\widetilde{T}_a^T &= \frac{-1}{|c|^2}(V^T)^{-1}\Sigma_0^\dagger V^{-1}\widetilde{T}_a V\Sigma_0 V^T \\ &= -\widetilde{T}_a\,. \end{aligned} \tag{8}$$

Likewise, one can use the same matrix V to define $\widetilde{X} \equiv VXV^{-1}$. By an analogous computation, $|c|^2 X^T = \Sigma_0^\dagger X \Sigma_0$, which implies that $\widetilde{X}_b^T = \widetilde{X}_b$. Moreover, since the generators of SU(N) are traceless and hermitian, it follows that the $\widetilde{X}_b$ are also traceless and real.

Thus, I have exhibited a similarity transformation that transforms the basis of the Lie algebra spanned by the T_a to one that is spanned by the $\widetilde{T}_a$. Since the $i\widetilde{T}_a$ are real antisymmetric matrices, one immediately recognizes this Lie algebra as that of SO(N). If an arbitrary element of the unbroken Lie algebra is exponentiated, it follows that $\exp i\theta_a T_a$ is related by a similarity transformation to $\exp i\theta_a \widetilde{T}_a$. The latter consists of arbitrary $N\times N$ real orthogonal matrices, which implies that the $\exp i\theta_a T_a$ constitutes an N

[1] Strictly speaking, the Takagi factorization yields a diagonal matrix with non-negative diagonal elements. If $c = |c|e^{2i\xi}$, one can obtain $\widetilde{V}\Sigma_0\widetilde{V}^T = |c| I_N$ by taking $\widetilde{V} = e^{i\xi}V$. However, this step is not necessary for the present argument.

dimensional representation that is equivalent to the N-dimensional defining representation of SO(N).

Finally, I note that from $|c|^2\, T_a^T = -\Sigma_0^\dagger T_a \Sigma_0$ and $|c|^2 X_b^T = \Sigma_0^\dagger X_b \Sigma_0$ it follows that $|c|^2\, T_a^T X_b^T = -\Sigma_0^\dagger T_a X_b \Sigma_0$ (since $\Sigma_0^\dagger \Sigma_0 = |c|^2 I_N$). Taking the trace yields Tr $T_a X_b = -$Tr $T_a X_b$, or equivalently Tr $T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full SU(N) Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b\}$ into $\{\widetilde{T}_a, \widetilde{X}_b\}$. I showed above that the $i\widetilde{T}_a$ are real antisymmetric matrices whereas the $\widetilde{X}_b$ are traceless real symmetric matrices. This implies that there are $\frac{1}{2}N(N-1)$ independent $\widetilde{T}_a$ and $\frac{1}{2}N(N+1)-1$ independent $\widetilde{X}_b$ (the 1 is subtracted to account for the extra condition that the $\widetilde{X}_b$ are traceless). The total number of SU(N) generators is therefore N^2-1 as expected.

The above results are easily verified explicitly for $N=3$. Consider for example a case in which

$$\Sigma_0 = c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \tag{9}$$

which clearly satisfies $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I$. Using the Gell-Mann matrices $\frac{1}{2}\lambda_a$ as the generators of SU(3), it is easy to check that $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$ implies that $T_a = c_1(\lambda_1 - \lambda_6) + c_2(\lambda_2 - \lambda_7) + c_3(\lambda_3 + \sqrt{3}\lambda_8)$, where $c_i \in \mathbb{R}$. That is, $\{\frac{1}{2}(\lambda_1 - \lambda_6)\,, \frac{1}{2}(\lambda_2 - \lambda_7)\,, \frac{1}{2}(\lambda_3 + \sqrt{3}\lambda_8)\}$ spans an SO(3) Lie subalgebra of the SU(3) Lie algebra. This matrix representation corresponds to the adjoint representation of an SU(2) subalgebra, which when exponentiated yields a representation equivalent to the defining representation of SO(3).

3. Symmetry breaking of SU($2N$) via a second-rank antisymmetric tensor

The case of symmetry breaking via an antisymmetric tensor exhibits many similar features. First, I shall consider the case of a global SU($2N$) symmetry group. Let Σ^{ab} be an antisymmetric second-rank tensor which transforms under SU($2N$) as $\Sigma \longrightarrow U\Sigma U^T$, where U is a $2N \times 2N$ unitary matrix with unit determinant. If Σ acquires a vacuum expectation value

$\langle\Sigma\rangle \equiv \Sigma_0$, then the SU($2N$) symmetry will be spontaneously broken. Writing $U = \exp i\theta_a T_a$ where the T_a are the unbroken generators, I again find

$$T_a\Sigma_0 + \Sigma_0 T_a^T = 0\,. \tag{10}$$

Thus, it is possible to find a basis for the traceless hermitian SU($2N$) generators given by $\{T_a, X_b\}$ such that the T_a satisfy eq. (10) and Tr $(T_a X_b) = 0$.

The unbroken subgroup H depends on the choice of Σ_0. Here, I shall consider the case of H=Sp($2N$), for which the most general form for Σ_0 is a complex antisymmetric $2N\times 2N$ matrix that satisfies $\Sigma_0^\dagger\Sigma_0 = \Sigma_0\Sigma_0^\dagger = |c|^2 I_{2N}$, where $c \in \mathbb{C}$ and I_{2N} is the $2N\times 2N$ unit matrix. This is summarized by the following theorem.

Theorem: Suppose that Σ_0 is a $2N\times 2N$ complex antisymmetric matrix that satisfies $\Sigma_0^\dagger\Sigma_0 = \Sigma_0\Sigma_0^\dagger = |c|^2 I_{2N}$ for some complex number c. Then, if the generators of SU($2N$) in the defining ($2N$-dimensional) representation are given by $\{T_a, X_b\}$, where the T_a and X_b are traceless hermitian $2N\times 2N$ matrices that satisfy:

$$T_a\Sigma_0 + \Sigma_0 T_a^T = 0\,, \tag{11}$$

$$X_b\Sigma_0 - \Sigma_0 X_b^T = 0\,, \tag{12}$$

then the T_a span an unbroken Sp($2N$) Lie subalgebra, while the X_b are the broken generators that span an SU($2N$)/Sp($2N$) homogeneous space. Furthermore, Tr $(T_a X_b) = 0$.

Proof: First, I show that if $\Sigma_0^\dagger\Sigma_0 = \Sigma_0\Sigma_0^\dagger = |c|^2 I_{2N}$ and $T_a\Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an Sp($2N$) Lie algebra. Note that these two conditions imply:

$$|c|^2\, T_a^T = -\Sigma_0^\dagger T_a \Sigma_0\,. \tag{13}$$

For any even-dimensional complex antisymmetric matrix M, there exists a unitary matrix W such that $WMW^T = \mathrm{diag}(\mathcal{J}_1\,, \mathcal{J}_2\,, \ldots\,, \mathcal{J}_n)$ is block diagonal, where each block is a 2×2 matrix of the form $\mathcal{J}_n \equiv \left(\begin{smallmatrix} 0 & z_n \\ -z_n & 0\end{smallmatrix}\right)$, where $z_n \in \mathbb{C}$ and the $|z_n|^2$ are the eigenvalues of $MM^\dagger$ (or $M^\dagger M$).[2] Applying this result to Σ_0, I note that the eigenvalues of $\Sigma_0\Sigma_0^\dagger$ are all degenerate and equal

[2]This result for complex antisymmetric matrices is the analog of the Takagi factorization for symmetric matrices [4]. Moreover, it is always possible to find a suitable choice for the unitary matrix W such that the z_i are real and non-negative. However, this step is not necessary for the present argument.

to $|c|^2$. Consider the matrix

$$J \equiv \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \tag{14}$$

where I_N is the $N \times N$ identity matrix. Since $JJ^\dagger = I_{2N}$, it follows that one can find unitary matrices W_1 and W_2 such that $W_1 \Sigma_0 W_1^T = cW_2 J W_2^T = \text{diag}(c\mathcal{J}, c\mathcal{J}, \ldots, c\mathcal{J})$, where

$$\mathcal{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{15}$$

That is, the factorization of Σ_0 and cJ both yield the same block diagonal matrix consisting of N identical 2×2 blocks consisting of $c\mathcal{J}$. Thus, there exists a unitary matrix $V = W_2^{-1} W_1$ such that

$$V \Sigma_0 V^T = c\, J\,. \tag{16}$$

The inverse of this result is $(V^T)^{-1} \Sigma_0^\dagger V^{-1} = -c^*\, J$ (since $\Sigma_0^\dagger = |c|^2 \Sigma_0^{-1}$ and $J^{-1} = -J$). I now define $\widetilde{T}_a \equiv V T_a V^{-1}$, where V is the unitary matrix appearing in eq. (16). Then, inserting this result into eq. (13), it follows that:

$$\begin{aligned} \widetilde{T}_a^T &= \frac{-1}{|c|^2} (V^T)^{-1} \Sigma_0^\dagger V^{-1} \widetilde{T}_a V \Sigma_0 V^T \\ &= J \widetilde{T}_a J\,. \end{aligned} \tag{17}$$

Likewise, one can use the same matrix V to define $\widetilde{X}_b \equiv V X_b V^{-1}$. By an analogous computation, $|c|^2 X^T = \Sigma_0^\dagger X \Sigma_0$, which implies that $\widetilde{X}_b^T = -J \widetilde{X}_b J$.

Thus, I have exhibited a similarity transformation that transforms the basis of the Lie algebra spanned by the T_a to one that is spanned by the $\widetilde{T}_a$. Since the $\widetilde{T}_a$ are traceless hermitian matrices[3] that satisfy $\widetilde{T}_a^T = J \widetilde{T}_a J$ [where J is defined by eq. (14)], one immediately recognizes this Lie algebra as that of Sp($2N$).[4] If an arbitrary element of the unbroken Lie algebra

[3]Since $J^2 = -I_{2N}$, it follows from $\widetilde{T}_a^T = J \widetilde{T}_a J$ that Tr $T_a = 0$. This implies that the group theory for the breaking of U($2N$) to Sp($2N$) would work in almost precisely the same way with one difference—the unbroken generators X_b would not be traceless.

[4]This is actually the unitary symplectic Lie algebra, which is the compact real form of the complex symplectic Lie algebra. Some books use the notation Sp(N) where I have used Sp($2N$). For more details, see ref. [5].

is exponentiated, it follows that $\exp i\theta_a T_a$ is related by a similarity transformation to $\exp i\theta_a \widetilde{T}_a$. The latter consists of arbitrary $2N \times 2N$ unitary symplectic matrices, which implies that the $\exp i\theta_a T_a$ constitutes an $2N$ dimensional representation that is equivalent to the $2N$-dimensional defining representation of Sp($2N$).

Finally, I note that from $|c|^2\, T_a^T = -\Sigma_0^\dagger T_a \Sigma_0$ and $|c|^2 X_b^T = \Sigma_0^\dagger X_b \Sigma_0$ it follows that $|c|^2\, T_a^T X_b^T = \Sigma_0^\dagger T_a X_b \Sigma_0$ (since $\Sigma_0^\dagger \Sigma_0 = |c|^2 I_{2N}$). Taking the trace yields Tr $T_a X_b = -$Tr $T_a X_b$, or equivalently Tr $T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full SU($2N$) Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b\}$ into $\{\widetilde{T}_a, \widetilde{X}_b\}$. Since these are all SU($2N$) generators, they are traceless hermitian matrices. Moreover, I showed above that the $\widetilde{T}_a$ satisfy $\widetilde{T}_a = J\widetilde{T}_a J$, whereas the $\widetilde{X}_b$ are traceless and satisfy $\widetilde{X}_b^T = -J\widetilde{X}_b J$. More explicitly,

$$\widetilde{T}_a = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix}\,, \qquad\qquad \widetilde{X}_b = \begin{pmatrix} C & D \\ D^\dagger & C^T \end{pmatrix}\,, \tag{18}$$

where A, B, C and D are $N \times N$ *complex* matrices such that A and C are hermitian, B is symmetric, D is antisymmetric and Tr $C = 0$. Thus, the number of independent real parameters that describe $\widetilde{T}_a$ corresponds to the number of parameters needed to define the hermitian matrix A and the complex symmetric matrix B, which is equal to $N^2 + N(N+1) = N(2N+1)$. Similarly, number of independent real parameters that describe $\widetilde{X}_b$ corresponds to the number of parameters needed to define the traceless hermitian matrix C and the complex antisymmetric matrix D, which is equal to $N^2-1+N(N+1) = N(2N-1)-1$. The total number of SU($2N$) generators is therefore $(2N)^2 - 1$ as expected.

4. Symmetry breaking of SU($2N+1$) via a second-rank antisymmetric tensor

For the case of spontaneous breaking of an SU($2N+1$) global symmetry by a second-rank antisymmetric tensor field, the analysis of the previous section requires some modification. In this case, Σ is a $(2N+1) \times (2N+1)$ matrix, which acquires a vacuum expectation value Σ_0. Once again, the unbroken generators T_a satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0\,. \tag{19}$$

In this case, one can find a basis for the traceless hermitian SU($2N+1$) generators given by $\{T_a, X_b, Y_c\}$ such that the T_a satisfy eq. (19) and Tr $(T_a X_b) =$ Tr $(T_a Y_c) =$ Tr $(X_b Y_c) = 0$.

I shall identify the unbroken subgroup H under the assumption that Σ_0 satisfies:

$$\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 \begin{pmatrix} I_{2N} & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{20}$$

where $c \in \mathbb{C}$. In this case, $H = \mathrm{Sp}(2N)$. This is summarized by the following theorem.

Theorem: Suppose that Σ_0 is a $(2N+1) \times (2N+1)$ complex antisymmetric matrix that satisfies eq. (20) for some complex number c. Then, if the generators of SU($2N+1$) in the defining [$(2N+1)$-dimensional] representation are given by $\{T_a, X_b, Y_c\}$, where the T_a, X_b and Y_c are traceless hermitian $(2N+1) \times (2N+1)$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0\,, \tag{21}$$

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0\,, \tag{22}$$

$$\Sigma_0^\dagger Y_c \Sigma_0 = 0\,, \tag{23}$$

then the T_a span an unbroken Sp($2N$) Lie subalgebra, while the $\{X_b, Y_c\}$ are the broken generators that span an SU($2N+1$)/Sp($2N$) homogeneous space. Furthermore, Tr $(T_a X_b) =$ Tr $(T_a Y_c) =$ Tr $(X_b Y_c) = 0$.

Proof: First, I show that if Σ_0 satisfies eq. (20) and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an Sp($2N$) Lie algebra. Here, I note that for any odd-dimensional complex antisymmetric matrix M, there exists a unitary matrix W such that $WMW^T = \mathrm{diag}(\mathcal{J}_1\,, \mathcal{J}_2\,, \ldots\,, \mathcal{J}_N\,, 0)$ where $\mathcal{J}_n \equiv \left(\begin{smallmatrix} 0 & z_n \\ -z_n & 0 \end{smallmatrix}\right)$, with $z_n \in \mathbb{C}$ and the $|z_n|^2$ are the eigenvalues of $MM^\dagger$ (or $M^\dagger M$) [4]. Introduce the $(2N+1) \times (2N+1)$ matrix:

$$K \equiv \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{24}$$

where J is the $2N \times 2N$ matrix given by eq. (14). The zeros shown above fill out the last row and the last column of the matrix K. Then applying the above factorization to the antisymmetric matrices Σ_0 and K, it follows that there exists a unitary matrix V such that

$$V\Sigma_0 V^T = cK\,. \tag{25}$$

Next, I multiply eq. (19) on the left by V and the right by V^T. Defining $\widetilde{T}_a \equiv VT_aV^{-1}$ as before and using eq. (25), one easily derives:

$$K\widetilde{T}_a^T = -\widetilde{T}_a K\,. \tag{26}$$

Using the fact that $\widetilde{T}_a$ is traceless and hermitian, eq. (26) has a unique solution:

$$\widetilde{T}_a = \begin{pmatrix} t_a & 0 \\ 0 & 0 \end{pmatrix}, \tag{27}$$

where t_a is an $2N \times 2N$ hermitian matrix that satisfies $t_a^T = Jt_aJ$. Thus, the $\widetilde{T}_a$ span an Sp($2N$) Lie subalgebra of the SU($2N+1$) Lie algebra.

Consider next the unbroken generators X_b and Y_c, which satisfy eqs. (22) and (23), and define $\widetilde{X}_b \equiv VX_bV^{-1}$ and $\widetilde{Y}_c \equiv VY_cV^{-1}$ Then eq. (25) implies that

$$K\widetilde{X}_b^T = \widetilde{X}_b K\,, \qquad K^\dagger \widetilde{Y}_c K = 0\,. \tag{28}$$

Using the fact that $\widetilde{X}_b$ and $\widetilde{Y}_c$ are traceless and hermitian, eq. (28) has a unique solution:

$$\widetilde{X}_b = \begin{pmatrix} x_b & 0 \\ 0 & -\mathrm{Tr}\ x_b \end{pmatrix}, \qquad \widetilde{Y}_c = \begin{pmatrix} 0 & y_c \\ y_c^\dagger & 0 \end{pmatrix}, \tag{29}$$

where x_b is an $2N \times 2N$ hermitian matrix that satisfies $x_b^T = -Jx_bJ$, and y_c is a complex $2N$-dimensional column vector. From the explicit forms above, it is easy to check that Tr $(\widetilde{T}_a\widetilde{X}_b) =$ Tr $(\widetilde{T}_a\widetilde{Y}_c) =$ Tr $(\widetilde{X}_b\widetilde{Y}_c) = 0$, which implies that Tr $(T_aX_b) =$ Tr $(T_aY_c) =$ Tr $(X_bY_c) = 0$.

To show that the $\{T_a, X_b, Y_c\}$ span the full SU($2N+1$) Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b, Y_c\}$ into $\{\widetilde{T}_a, \widetilde{X}_b, \widetilde{Y}_c\}$. Since these are all SU($2N+1$) generators, they are traceless hermitian matrices. Moreover, I showed above that the $\widetilde{T}_a$, $\widetilde{X}_b$ and $\widetilde{Y}_c$ are given by eqs. (27) and (29), where $t_a^T = Jt_aJ$ and $x_b^T = -Jx_bJ$ are $2N \times 2N$ hermitian matrices and y_c is a complex $2N$-dimensional vector. However, in contrast to $\widetilde{X}_b$ in the last section, x_b is *not* traceless. Following the analysis at the end of the previous section (but with Tr $x_b \neq 0$), it follows that the number of independent real parameters that describe $\widetilde{T}_a$ and $\widetilde{X}_b$ is given by $N(2N+1)$ and $N(2N-1)$, respectively. Adding this to the $4N$ parameters that describe $\widetilde{Y}_c$ yields a total number of SU$(2N+1)$ generators equal to $(2N+1)^2-1$ as expected.

Finally, it is interesting to note that the generators of the type Y_c do not appear in the breaking of SU$(2N)$ to Sp$(2N)$ described in the previous section. This is easy to see by noting that in the previous section, $\Sigma_0^\dagger\Sigma_0 = |c|^2 I_{2N}$ implies that Σ_0^{-1} exists. Thus, in this case $\Sigma_0^\dagger Y_c \Sigma_0 = 0$ would imply that $Y_c = 0$. In the case of SU$(2N+1)$ breaking, since Σ_0 is an odd-dimensional antisymmetric matrix, it follows that $\det\Sigma_0 = 0$. Thus, Σ_0^{-1} does not exist and an non-trivial solution for Y_c can arise, as we have explicitly shown above.

5. Symmetry breaking of SO($2N$) and SO($2N+1$) via a second-rank antisymmetric tensor

The case of spontaneous breaking of SO$(2N)$ or SO$(2N+1)$ to U(N) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is very similar to the corresponding breaking of SU$(2N)$ or SU$(2N+1)$ considered in the previous two sections. Thus, we provide a few details here. In the case of SO$(2N)$ the relevant theorem is as follows.

Theorem: Suppose that Σ_0 is a $2N\times 2N$ real antisymmetric matrix that satisfies $\Sigma_0^T\Sigma_0 = \Sigma_0\Sigma_0^T = c^2 I_{2N}$ for some real number c. Then, if the generators of SO$(2N)$ in the defining ($2N$-dimensional) representation are given by $\{T_a, X_b\}$, where the iT_a and iX_b are real antisymmetric $2N\times 2N$ matrices that satisfy:

$$T_a\Sigma_0 + \Sigma_0 T_a^T = 0\,, \tag{30}$$
$$X_b\Sigma_0 - \Sigma_0 X_b^T = 0\,, \tag{31}$$

then the T_a span an unbroken U(N) Lie subalgebra, while the X_b are the broken generators that span an SO$(2N)$/U(N) homogeneous space. Furthermore, Tr $(T_a X_b) = 0$.

Proof: First, I show that if $\Sigma_0^T\Sigma_0 = \Sigma_0\Sigma_0^T = c^2 I_{2N}$ and $T_a\Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an U(N) Lie subalgebra. Note that these two conditions imply:

$$c^2\, T_a^T = -\Sigma_0^T T_a \Sigma_0\,. \tag{32}$$

For any even-dimensional real antisymmetric matrix M, there exists a real orthogonal matrix W such that $WMW^T = \mathrm{diag}(\mathcal{J}_1\,, \mathcal{J}_2\,, \ldots\,, \mathcal{J}_n)$ is block diagonal, where each block is a 2×2 matrix of the form $\mathcal{J}_n \equiv \left(\begin{smallmatrix} 0 & z_n \\ -z_n & 0\end{smallmatrix}\right)$, where $z_n \in \mathbb{R}$ and the z_n^2 are the eigenvalues of MM^T (or M^TM).[5] Applying this result to Σ_0, note that the eigenvalues of $\Sigma_0\Sigma_0^T$ are all degenerate and equal to c^2. Moreover, since the matrix J [eq. (14)] satisfies $JJ^T = I_{2N}$, it follows that one can find real orthogonal matrices W_1 and W_2 such that $W_1\Sigma_0 W_1^T = cW_2 J W_2^T = \mathrm{diag}(c\mathcal{J}, c\mathcal{J}, \ldots, c\mathcal{J})$, where $\mathcal{J}$ is defined in eq. (15). That is, the factorization of Σ_0 and cJ both yield the same block diagonal matrix consisting of N identical 2×2 blocks consisting of $c\mathcal{J}$. Thus, there exists a real orthogonal matrix $V = W_2^{-1}W_1$ such that $V\Sigma_0 V^T = c\,J$. The inverse of this result is $V\Sigma_0^T V^T = -c\,J$ (since $J^T = -J$). I now define $\widetilde{T}_a \equiv VT_aV^T$. Then eq. (32) implies that

$$\widetilde{T}_a^T = \frac{-1}{c^2} V\Sigma_0^T V^T \widetilde{T}_a V\Sigma_0 V^T = J\widetilde{T}_a J\,. \tag{33}$$

Likewise, one can use the same matrix V to define $\widetilde{X}_b \equiv VX_bV^T$. By an analogous computation, $c^2X^T = \Sigma_0^T X\Sigma_0$, which implies that $\widetilde{X}_b^T = -J\widetilde{X}_b J$.

Recall that that T_a and X_b are both antisymmetric $2N\times 2N$ matrices. Then, $\widetilde{T}_a \equiv VT_aV^T$ and $\widetilde{X}_a \equiv VX_aV^T$ are also antisymmetric. Hence, it follows that

$$\widetilde{T}_a = -J\widetilde{T}_a J\,, \qquad \widetilde{X}_a = J\widetilde{X}_a J\,. \tag{34}$$

Using the explicit form for J, eq. (34) implies that T_a and X_b take the following block form:

$$i\,\widetilde{T}_a = \begin{pmatrix} A & B \\ -B & A\end{pmatrix}\,, \qquad\qquad i\widetilde{X}_b = \begin{pmatrix} C & D \\ D & -C\end{pmatrix}\,, \tag{35}$$

where A, B, C and D are $N\times N$ real matrices such that A, C and D are antisymmetric and B is symmetric. Thus, I have exhibited a similarity transformation (note that $V^T = V^{-1}$) that transforms the basis of the Lie algebra spanned by the T_a to one that is spanned by the $\widetilde{T}_a$. Moreover, consider the isomorphism that maps $i\,\widetilde{T}_a$ given in eq. (35) to the $N\times N$

[5] This result for real antisymmetric matrices is the analog of the corresponding factorization of complex antisymmetric matrices quoted in Section 3.

matrix $A+iB$. Since $(A+iB)^\dagger=(A-iB)^T=-(A+iB)$, we see that the $A+iB$ are anti-hermitian generators (which are not generally traceless) that span a U(N) subalgebra of the SO($2N$). We can check the number of unbroken generators by counting the number of degrees of freedom of one real antisymmetric and one real symmetric matrix: $\frac{1}{2}N(N-1)+\frac{1}{2}N(N+1)=N^2$, as expected.

Finally, I note that from $c^2\,T_a^T=-\Sigma_0^T T_a\Sigma_0$ and $c^2 X_b^T=\Sigma_0^T X_b\Sigma_0$ it follows that $c^2\,T_a^T X_b^T=\Sigma_0^T T_a X_b\Sigma_0$ (since $\Sigma_0^T\Sigma_0=c^2 I_{2N}$). Taking the trace yields Tr $T_a X_b=-$Tr $T_a X_b$, or equivalently Tr $T_a X_b=0$. To show that the $\{T_a, X_b\}$ span the full SO($2N$) Lie algebra, we note that there are N^2 unbroken generators and $N(N-1)$ broken generators (corresponding to the number of parameters describing two real antisymmetric matrices [see eq. (35)]). Thus, the total number of generators is $N(2N-1)$ which matches the total number of SO($2N$) generators.

Finally, we turn to the case of SO($2N+1$) breaking. In this case, we will make use of the fact that for any odd-dimensional real antisymmetric matrix M, there exists a real orthogonal matrix W such that $WMW^T=\mathrm{diag}(\mathcal{J}_1\,,\mathcal{J}_2\,,\ldots\,,\mathcal{J}_N\,,0)$ where $\mathcal{J}_n\equiv\left(\begin{smallmatrix}0 & z_n\\ -z_n & 0\end{smallmatrix}\right)$, with $z_n\in\mathbb{R}$ and the z_n^2 are the eigenvalues of MM^T (or $M^T M$). The relevant theorem for case of SO($2N+1$) is as follows.

Theorem: Suppose that Σ_0 is a $(2N+1)\times(2N+1)$ real antisymmetric matrix that satisfies

$$\Sigma_0^T\Sigma_0=\Sigma_0\Sigma_0^T=c^2\begin{pmatrix} I_{2N} & 0\\ 0 & 0\end{pmatrix}\,, \tag{36}$$

where $c\in\mathbb{R}$. Then, if the generators of SO($2N+1$) in the defining [$(2N+1)$-dimensional] representation are given by $\{T_a, X_b, Y_c\}$, where the iT_a, iX_b and iY_c are real antisymmetric $(2N+1)\times(2N+1)$ matrices that satisfy:

$$T_a\Sigma_0+\Sigma_0 T_a^T=0\,, \tag{37}$$

$$X_b\Sigma_0-\Sigma_0 X_b^T=0\,, \tag{38}$$

$$\Sigma_0^T Y_c\Sigma_0=0\,, \tag{39}$$

then the T_a span an unbroken U(N) Lie subalgebra, while the $\{X_b, Y_c\}$ are the broken generators that span an SO($2N+1$)/U(N) homogeneous space. Furthermore, Tr $(T_a X_b)=$ Tr $(T_a Y_c)=$ Tr $(X_b Y_c)=0$.

Proof: Here, I shall only sketch the modifications to the proof given in Section 4. Again, we easily derive $K\widetilde{T}_a^T=-\widetilde{T}_a K$ [where K is defined in eq. (24)]. In the present case, we use the fact that $\widetilde{T}_a$ is antisymmetric to conclude that $K\widetilde{T}_a = \widetilde{T}_a K$. That is, we may write the unbroken generators, $\widetilde{T}_a$, in the form of eq. (27) where $t_a = -Jt_aJ$ and the it_a are $2N \times 2N$ real antisymmetric matrices. Using the results previously obtained, it follows that the $\widetilde{T}_a$ span a U(N) subalgebra. Likewise, the broken generators satisfy: $K\widetilde{X}_a = -\widetilde{X}_a K$ and $K^T\widetilde{Y}_c K = 0$. The antisymmetry of $\widetilde{X}_a$ and $\widetilde{Y}_a$ implies

$$i\widetilde{X}_b = \begin{pmatrix} x_b & 0 \\ 0 & 0 \end{pmatrix}, \qquad i\widetilde{Y}_c = \begin{pmatrix} 0 & y_c \\ -y_c^T & 0 \end{pmatrix}, \tag{40}$$

where x_b is an $2N \times 2N$ real antisymmetric matrix that satisfies $x_b = Jx_bJ$, and y_c is a real $2N$-dimensional column vector. From the explicit forms above, it is easy to check that Tr $(\widetilde{T}_a\widetilde{X}_b) =$ Tr $(\widetilde{T}_a\widetilde{Y}_c) =$ Tr $(\widetilde{X}_b\widetilde{Y}_c) = 0$, which implies that Tr $(T_aX_b) =$ Tr $(T_aY_c) =$ Tr $(X_bY_c) = 0$.

Finally, we count the number of SO($2N+1$) generators $\{\widetilde{T}_a, \widetilde{X}_b, \widetilde{Y}_c\}$. There are N^2 unbroken generators and $N(N+1)$ degrees of freedom associated with $\widetilde{X}_b$ as in the case of SO($2N$) breaking. Finally, adding in the $2N$ parameters that describes $\widetilde{Y}_c$ yields a total number of SO($2N+1$) generators equal to $N(2N+1)$ as expected.

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