The tensor spherical harmonics

Howard E. Haber

Santa Cruz Institute for Particle Physics

University of California, Santa Cruz, CA 95064, USA

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Abstract

In these notes, the vector spherical harmonics are presented in a general context, which provides for a better understanding of their origin (and allows for further generalizations). The formalism of addition of angular momentum in quantum mechanics plays a key role in this general treatment. In the final section of these notes, the application of vector spherical harmonics to the electromagnetic multipole radiation fields is exhibited.

1 The Clebsch-Gordon coefficients

Consider a system with orbital angular momentum $\vec{\boldsymbol{L}}$ and spin angular momentum $\vec{\boldsymbol{S}}$. The total angular momentum of the system is denoted by $\vec{\boldsymbol{J}} = \vec{\boldsymbol{L}} + \vec{\boldsymbol{S}}$. Clebsch Gordon coefficients allow us to express the total angular momentum basis $|j\,m\,;\,\ell\,s\rangle$ in terms of the direct product basis, $|\ell\,m_\ell\,;\,s\,m_s\rangle \equiv |\ell\,m_\ell\rangle \otimes |s\,m_s\rangle$,

$$|j\,m\,;\,\ell\,s\rangle = \sum_{m_\ell=-\ell}^{\ell} \sum_{m_s=-s}^{s} \langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\,;\,\ell\,s\rangle |\ell\,m_\ell\,;\,s\,m_s\rangle\,. \tag{1}$$

The Clebsch-Gordon coefficient is often denoted by (cf. pp. 412–415 of Ref. [1]):

$$\langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\rangle \equiv \langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\,;\,\ell\,s\rangle\,,$$

since including $\ell\,s$ in $|j\,m\,;\,\ell\,s\rangle$ on the right hand side above is redundant information.

One important property of the Clebsch-Gordon coefficients is

$$\langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\rangle = \delta_{m\,,\,m_\ell+m_s} \langle \ell\,m_\ell\,;\,s\;m_s\,|\,j\;m_\ell+m_s\rangle\,. \tag{2}$$

In particular, if $m \neq m_\ell + m_s$ then the corresponding Clebsch-Gordon coefficient must vanish. This is simply a consequence of $J_z = L_z + S_z$. Likewise, $|\ell - s| \leq j \leq \ell + s$ (where $2j$, ℓ and $2s$ are non-negative integers), otherwise the corresponding Clebsch-Gordon coefficients vanish.

Recall that in the coordinate representation, the angular moment operator in quantum mechanics is a differential operator given by

$$\vec{\boldsymbol{L}} = -i\hbar\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}}\,.$$

The spherical harmonics, $Y_{\ell m_\ell}(\theta,\phi)$ are simultaneous eigenstates of $\vec{\boldsymbol{L}}^2$ and L_z,

$$\vec{\boldsymbol{L}}^2\,Y_{\ell m_\ell}(\theta,\phi) = \hbar^2\ell(\ell+1)\,Y_{\ell m_\ell}(\theta,\phi)\,, \qquad L_z\,Y_{\ell m_\ell}(\theta,\phi) = \hbar m_\ell\,Y_{\ell m_\ell}(\theta,\phi)\,. \tag{3}$$

We can generalize these results to systems with non-zero spin. First, we define $\chi_{s\,m_s}$ to be the simultaneous eigenstates of $\vec{\boldsymbol{S}}^2$ and S_z ,

$$\vec{\boldsymbol{S}}^2\chi_{s\,m_s} = \hbar^2 s(s+1)\chi_{s\,m_s}\,, \qquad S_z\chi_{s\,m_s} = \hbar m_s\chi_{s\,m_s}\,.$$

The direct product basis in the coordinate representation is given by $Y_{\ell m_\ell}(\theta,\phi)\chi_{s\,m_s}$.

2 Definition of the tensor spherical harmonics

In the coordinate representation, the total angular momentum basis consists of simultaneous eigenstates of $\vec{\boldsymbol{J}}^2$, J_z , $\vec{\boldsymbol{L}}^2$, $\vec{\boldsymbol{S}}^2$. These are the *tensor spherical harmonics*, which satisfy,

$$
\begin{aligned}
&\vec{\boldsymbol{J}}^2\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)=\hbar^2 j(j+1)\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,, \qquad && J_z\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)=\hbar m\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,,\\
&\vec{\boldsymbol{L}}^2\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)=\hbar^2 \ell(\ell+1)\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,, \qquad && \vec{\boldsymbol{S}}^2\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)=\hbar^2 s(s+1)\,\mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,.
\end{aligned}
$$

As a consequence of eq. (1), the tensor spherical harmonics are defined by

$$
\begin{aligned}
\mathcal{Y}^{\ell s}_{jm}(\theta,\phi) &= \sum_{m_\ell=-\ell}^{\ell}\,\sum_{m_s=-s}^{s}\langle \ell\, m_\ell\,;\, s\, m_s\,|\, j\, m\rangle Y_{\ell m_\ell}(\theta,\phi)\chi_{s\,m_s}\\
&= \sum_{m_s=-s}^{s}\langle \ell\,,\, m-m_s\,;\, s\, m_s\,|\, j\, m\rangle Y_{\ell\,,\,m-m_s}(\theta,\phi)\chi_{s\,m_s}\,,
\end{aligned} \tag{4}
$$

where the second line follows from the first line above since the Clebsch-Gordon coefficient above vanishes unless $m = m_\ell + m_s$.

The general expressions for the Clebsch-Gordon coefficients in terms of j, m_ℓ, ℓ, s and m_s are very complicated to write down. Nevertheless, the explicit expressions in the simplest cases of $s = 1/2$ and $s = 1$ are manageable. Thus, we shall exhibit these two special cases below.

3 The spinor spherical harmonics

For spin $s = 1/2$, the possible values of j are $j = \ell + \frac{1}{2}$ and $\ell - \frac{1}{2}$, for $\ell = 1, 2, 3, \ldots$. If $\ell = 0$ then only $j = \frac{1}{2}$ is possible (and the last row of Table 1 should be omitted). The corresponding table of Clebsch-Gordon coefficients is exhibited in Table 1.

Table 1: the Clebsch-Gordon coefficients, $\langle \ell\, m - m_s\,;\, \frac{1}{2}\, m_s\,|\, jm\rangle$.

j	$m_s = \frac{1}{2}$	$m_s = -\frac{1}{2}$
$\ell + \frac{1}{2}$	$\left(\dfrac{\ell + m + \frac{1}{2}}{2\ell + 1}\right)^{1/2}$	$\left(\dfrac{\ell - m + \frac{1}{2}}{2\ell + 1}\right)^{1/2}$
$\ell - \frac{1}{2}$	$-\left(\dfrac{\ell - m + \frac{1}{2}}{2\ell + 1}\right)^{1/2}$	$\left(\dfrac{\ell + m + \frac{1}{2}}{2\ell + 1}\right)^{1/2}$

Comparing with eq. (4), the entries in Table 1 are equivalent to the following result:

$$
\left|j=\ell\pm\tfrac{1}{2}\, m\right\rangle = \frac{1}{\sqrt{2\ell+1}}\left[\pm\sqrt{\ell+\tfrac{1}{2}\pm m}\,\left|\ell\, m-\tfrac{1}{2}\,;\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \sqrt{\ell+\tfrac{1}{2}\mp m}\,\left|\ell\, m+\tfrac{1}{2}\,;\,\tfrac{1}{2}\; -\tfrac{1}{2}\right\rangle\right].
$$

We can represent $|\frac{1}{2}\,\frac{1}{2}\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|\frac{1}{2}\,-\frac{1}{2}\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Then in the coordinate representation, the *spin spherical harmonics* are given by

$$\mathcal{Y}^{\ell\,\frac{1}{2}}_{j=\ell\pm\frac{1}{2}\,,\,m}(\theta,\phi) \equiv \langle\theta\,\phi\,|\,j=\ell\pm\tfrac{1}{2}\,,\,m\rangle = \frac{1}{\sqrt{2\ell+1}}\begin{pmatrix} \pm\sqrt{\ell\pm m+\frac{1}{2}}\;Y_{\ell,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\ell\mp m+\frac{1}{2}}\;Y_{\ell,m+\frac{1}{2}}(\theta,\phi)\end{pmatrix}. \tag{5}$$

If $\ell = 0$, there is only one spin spherical harmonic,

$$\mathcal{Y}^{0\,\frac{1}{2}}_{j=\frac{1}{2}\,,\,m}(\theta,\phi) \equiv \langle\theta\,\phi\,|\,j=\tfrac{1}{2}\,,\,m\rangle = \frac{1}{\sqrt{2\ell+1}}\begin{pmatrix} \sqrt{\frac{1}{2}+m}\;Y_{0,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{1}{2}-m}\;Y_{0,m+\frac{1}{2}}(\theta,\phi)\end{pmatrix}. \tag{6}$$

Note that when $m = \frac{1}{2}$ the lower component of eq. (6) vanishes and when $m = -\frac{1}{2}$ the upper component of eq. (6) vanishes. In both cases, the non-vanishing component is proportional to $Y_{00}(\theta,\phi) = 1/\sqrt{4\pi}$.

4 The vector spherical harmonics

For spin $s = 1$, the possible values of j are $j = \ell+1\,,\,\ell\,,\,\ell-1$ for $\ell = 1, 2, 3, \ldots$. If $\ell = 0$ then only $j = 1$ is possible (and the last two rows exhibited in Table 2 should be omitted). The corresponding table of Clebsch-Gordon coefficients is exhibited in Table 2.

Table 2: the Clebsch-Gordon coefficients, $\langle \ell\, m-m_s\,;\,1\,m_s\,|\,jm\rangle$.

j	$m_s = 1$	$m_s = 0$	$m_s = -1$
$\ell+1$	$\left[\dfrac{(\ell+m)(\ell+m+1)}{(2\ell+1)(2\ell+2)}\right]^{1/2}$	$\left[\dfrac{(\ell-m+1)(\ell+m+1)}{(\ell+1)(2\ell+1)}\right]^{1/2}$	$\left[\dfrac{(\ell-m)(\ell-m+1)}{(2\ell+1)(2\ell+2)}\right]^{1/2}$
ℓ	$-\left[\dfrac{(\ell+m)(\ell-m+1)}{2\ell(\ell+1)}\right]^{1/2}$	$\dfrac{m}{\sqrt{\ell(\ell+1)}}$	$\left[\dfrac{(\ell-m)(\ell+m+1)}{2\ell(\ell+1)}\right]^{1/2}$
$\ell-1$	$\left[\dfrac{(\ell-m)(\ell-m+1)}{2\ell(2\ell+1)}\right]^{1/2}$	$-\left[\dfrac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}\right]^{1/2}$	$\left[\dfrac{(\ell+m)(\ell+m+1)}{2\ell(2\ell+1)}\right]^{1/2}$

Using a spherical basis, we can represent $|1\,1\rangle = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $|1\,0\rangle = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$ and $|1\,-1\rangle = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. With respect to this basis, we can explicitly write out the three *vector spherical harmonics*, $\mathcal{Y}^{\ell\,1}_{j=\ell\pm1\,,\,m}(\theta,\phi)$ and $\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)$. For example, if $\ell \neq 0$ then,

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)=\begin{pmatrix} -\left[\dfrac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}\right]^{1/2} Y_{\ell,m-1}(\theta,\phi) \\ \dfrac{m}{\sqrt{\ell(\ell+1)}}\, Y_{\ell m}(\theta,\phi) \\ \left[\dfrac{(\ell+m+1)(\ell-m)}{2\ell(\ell+1)}\right]^{1/2} Y_{\ell,m+1}(\theta,\phi)\end{pmatrix}.$$

The other two vector spherical harmonics can be written out in a similar fashion. If $\ell=0$ then $\mathcal{Y}^{\ell\,1}_{j=\ell+1\,,\,m}(\theta,\phi)$ is the only surviving vector spherical harmonic.

It is instructive to work in a Cartesian basis, where the χ_{1,m_s} are eigenvectors of $\mathcal{S}_3$, and the spin-1 spin matrices are given by $\hbar\vec{\boldsymbol{\mathcal{S}}}$, where $(\mathcal{S}_k)_{ij}=-i\epsilon_{ijk}$. In particular,

$$\mathcal{S}_3=\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}.$$

and $\mathcal{S}_3\chi_{1,m_s}=m_s\chi_{1,m_s}$. This yields the orthonormal eigenvectors,

$$\chi_{1,\pm 1}=\frac{1}{\sqrt{2}}\begin{pmatrix} \mp 1 \\ -i \\ 0\end{pmatrix}, \qquad \chi_{1,0}=\begin{pmatrix} 0 \\ 0 \\ 1\end{pmatrix}. \tag{7}$$

where the arbitrary overall phase factors are conventionally chosen to be unity. As an example, in the Cartesian basis,

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)=\frac{1}{2\sqrt{\ell(\ell+1)}}\begin{pmatrix} [(\ell-m+1)(\ell+m)]^{1/2}\, Y_{\ell,m-1}(\theta,\phi)+[(\ell+m+1)(\ell-m)]^{1/2}\, Y_{\ell,m+1}(\theta,\phi) \\ i\,[(\ell-m+1)(\ell+m)]^{1/2}\, Y_{\ell,m-1}(\theta,\phi)-i\,[(\ell+m+1)(\ell-m)]^{1/2}\, Y_{\ell,m+1}(\theta,\phi) \\ 2mY_{\ell m}(\theta,\phi)\end{pmatrix}. \tag{8}$$

This is a vector with respect to the basis $\{\hat{\boldsymbol{x}}\,,\,\hat{\boldsymbol{y}}\,,\,\hat{\boldsymbol{z}}\}$. It is convenient to rewrite eq. (8) in terms of the basis $\{\hat{\boldsymbol{r}}\,,\,\hat{\boldsymbol{\theta}}\,,\,\hat{\boldsymbol{\phi}}\}$ using

$$\begin{aligned}
\hat{\boldsymbol{x}}&=\hat{\boldsymbol{r}}\sin\theta\cos\phi+\hat{\boldsymbol{\theta}}\cos\theta\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi\,,\\
\hat{\boldsymbol{y}}&=\hat{\boldsymbol{r}}\sin\theta\sin\phi+\hat{\boldsymbol{\theta}}\cos\theta\sin\phi+\hat{\boldsymbol{\phi}}\cos\phi\,,\\
\hat{\boldsymbol{z}}&=\hat{\boldsymbol{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\,.
\end{aligned}$$

We can then greatly simplify the resulting expression for $\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)$ by employing the recursion relation,

$$\begin{aligned}
-2m\cos\theta\, Y_{\ell m}(\theta,\phi)=\sin\theta\Big\{&[(\ell+m+1)(\ell-m)]^{1/2}\, e^{-i\phi}\, Y_{\ell,m+1}(\theta,\phi)\\
&+[(\ell-m+1)(\ell+m)]^{1/2}\, e^{i\phi}\, Y_{\ell,m-1}(\theta,\phi)\Big\}\,,
\end{aligned}$$

and the following two differential relations,

$$\frac{\partial}{\partial\phi}\,Y_{\ell m}(\theta,\phi) = imY_{\ell m}(\theta,\phi)\,,$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\phi) = \tfrac{1}{2}\left[(\ell+m+1)(\ell-m)\right]^{1/2} e^{-i\phi}\,Y_{\ell,m+1}(\theta,\phi)\\ -\tfrac{1}{2}\left[(\ell-m+1)(\ell+m)\right]^{1/2} e^{i\phi}\,Y_{\ell,m-1}(\theta,\phi)\,.\end{aligned}$$

Following a straightforward but tedious computation, the end result is:

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi) = \frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\,\frac{\partial}{\partial\phi}\,Y_{\ell m}(\theta,\phi) - \hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\phi)\right]\,.$$

At this point, one should recognize the differential operator $\vec{\boldsymbol{L}}$ expressed in the $\{\hat{\boldsymbol{r}}\,,\,\hat{\boldsymbol{\theta}}\,,\,\hat{\boldsymbol{\phi}}\}$ basis,

$$\vec{\boldsymbol{L}} = -i\hbar\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}} = i\hbar\left[\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\,\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\right]\,. \tag{9}$$

Hence, we end up with

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi) = \frac{1}{\sqrt{\hbar^2\ell(\ell+1)}}\,\vec{\boldsymbol{L}}\,Y_{\ell m}(\theta,\phi)\,,\qquad \text{for } \ell\neq 0\,. \tag{10}$$

This is the vector spherical harmonic,

$$\boxed{\vec{\boldsymbol{X}}_{\ell m}(\theta,\phi) = \frac{-i}{\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\phi)} \tag{11}$$

employed by J.D. Jackson in Ref. [14]. Note that eq. (11) satisfies the following relation:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{X}}_{\ell m}(\theta,\phi) = 0\,. \tag{12}$$

To prove eq. (12), we compute

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\phi)\big] = \epsilon_{ijk}\partial_i\big[x_j\partial_k\,Y_{\ell m}(\theta,\phi)\big] = 0\,, \tag{13}$$

where there is an implicit sum over pairs of repeated indices and $\partial_i \equiv \partial/\partial x_i$. The final result of eq. (13) is obtained after noting that $\partial_i x_j = \delta_{ij}$ and employing $\epsilon_{ijk}\delta_{ij} = 0$ and $\epsilon_{ijk}\partial_i\partial_k = 0$. These last two relations follow since a symmetric tensor summed against an antisymmetric tensor vanishes.

Using the same methods, one can derive the following expressions for the other two vector spherical harmonics,

$$\mathcal{Y}^{\ell\,1}_{j=\ell-1\,,\,m}(\theta,\phi) = \frac{-1}{\sqrt{(j+1)(2j+1)}}\left[(j+1)\hat{\boldsymbol{n}} - r\vec{\boldsymbol{\nabla}}\right]Y_{jm}(\theta,\phi)\,,\qquad \text{for } \ell\neq 0\,, \tag{14}$$

$$\mathcal{Y}^{\ell\,1}_{j=\ell+1\,,\,m}(\theta,\phi) = \frac{1}{\sqrt{j(2j+1)}}\left[j\hat{\boldsymbol{n}} + r\vec{\boldsymbol{\nabla}}\right]Y_{jm}(\theta,\phi)\,, \tag{15}$$

where $\hat{\boldsymbol{n}} \equiv \hat{\boldsymbol{r}} = \vec{\boldsymbol{x}}/r$. That is, the three independent normalized vector spherical harmonics can be chosen as:

$$\left\{ \frac{-ir}{\sqrt{j(j+1)}}\, \hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi)\,, \quad \frac{r}{\sqrt{j(j+1)}} \vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi)\,, \quad \hat{\boldsymbol{n}}\, Y_{jm}(\theta,\phi) \right\}. \tag{16}$$

It is often convenient to rewrite

$$r\, \vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi) = -r[\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}) - \vec{\boldsymbol{\nabla}}]Y_{jm}(\theta,\phi) = -r\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}})Y_{jm}(\theta,\phi)\,, \tag{17}$$

after noting that

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi) = \frac{\partial Y_{jm}(\theta,\phi)}{\partial r} = 0\,.$$

Then, the list of the three independent normalized vector spherical harmonics takes the following form:

$$\left\{ \frac{-ir}{\sqrt{j(j+1)}}\, \hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi)\,, \quad \frac{-r}{\sqrt{j(j+1)}}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}})Y_{jm}(\theta,\phi)\,, \quad \hat{\boldsymbol{n}}\, Y_{jm}(\theta,\phi) \right\}. \tag{18}$$

The first two vector spherical harmonics, $\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi)$ and $\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}})Y_{jm}(\theta,\phi)$, are transverse (i.e., perpendicular to $\hat{\boldsymbol{n}}$), whereas the third vector spherical harmonic, $\hat{\boldsymbol{n}}\, Y_{jm}(\theta,\phi)$, is longitudinal (i.e., parallel to $\hat{\boldsymbol{n}}$). In particular, when employed in the multipole expansion of the (transverse) electric and magnetic radiation fields in the radiation zone, only the first two vector spherical harmonics of eq. (18) appear (cf. Section 6).

Note that the second and third vector spherical harmonics listed in eq. (16) [or eq. (18)], $r\vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi)$ and $\hat{\boldsymbol{n}}\, Y_{jm}(\theta,\phi)$, are not eigenstates of $\vec{\boldsymbol{L}}^2$ since they consist of linear combinations of states with $\ell = j \pm 1$ [which can be explicitly derived by inverting eqs. (14) and (15)]. This observation will be confirmed in eqs. (35) and (44) below.

The algebraic steps involved in establishing eqs. (10)–(15) are straightforward but tedious. A more streamlined approach to the derivation of these results is given in the next section.

5 The vector spherical harmonics revisited

Since $Y_{\ell m}(\hat{\boldsymbol{n}})$ is a spherical tensor of rank-ℓ, and $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$, $\vec{\boldsymbol{L}} \equiv -i\hbar\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}}$, and $r\vec{\boldsymbol{\nabla}}$ are vector operators, it is not surprising that the vector spherical harmonics are linear combinations of the quantities given in eq. (16). It is instructive to derive this result directly. For convenience, we shall adopt the notation of Ref. [12] by denoting the vector spherical harmonics in this section by

$$\vec{\boldsymbol{Y}}_{j\ell m}(\hat{\boldsymbol{n}}) \equiv \mathcal{Y}^{\ell 1}_{jm}(\theta,\phi)\,, \quad \text{for } j = \ell+1\,,\ \ell\,,\ \ell-1\,, \tag{19}$$

where $\hat{\boldsymbol{n}}$ is a unit vector with polar angle θ and azimuthal angle ϕ.

First, we recall that (e.g., see eq. (12.5.20) of Ref. [1]):

$$L_\pm|\ell\, m\rangle = \hbar\left[(\ell \mp m)(\ell \pm m+1)\right]^{1/2}|\ell\, m\pm 1\rangle\,, \qquad L_z|\ell\, m\rangle = \hbar m|\ell\, m\rangle\,, \tag{20}$$

where $L_\pm \equiv L_x \pm iL_y$. The spherical components of $\vec{\boldsymbol{L}}$ are L_q $(q = +1, 0, -1)$ where

$$L_{\pm 1} \equiv \mp\frac{L_\pm}{\sqrt{2}} = \mp\frac{1}{\sqrt{2}}\left(L_x \pm iL_y\right)\,, \qquad L_0 \equiv L_z\,.$$

Using the Clebsch-Gordon coefficients given in Table 2, it follows that

$$L_q|\ell\, m\rangle = \hbar(-1)^q\sqrt{\ell(\ell+1)}\,\langle \ell\,,\, m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,|\ell\,,\,m+q\rangle\,. \tag{21}$$

In the coordinate representation, eq. (21) is equivalent to

$$L_q\,Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar(-1)^q\sqrt{\ell(\ell+1)}\,\langle \ell\,,\, m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,Y_{\ell\,,\,m+q}(\hat{\boldsymbol{n}})\,. \tag{22}$$

It is convenient to introduce a set of spherical basis vectors,

$$\hat{\boldsymbol{e}}_{\pm 1} \equiv \mp\frac{1}{\sqrt{2}}\,(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})\;, \qquad\qquad \hat{\boldsymbol{e}}_0 \equiv \hat{\boldsymbol{z}}\,. \tag{23}$$

It is not surprising that $\hat{\boldsymbol{e}}_q = \chi_{1,q}$ [cf. eq. (7)]. One can check that

$$\vec{\boldsymbol{L}} = L_x\hat{\boldsymbol{x}} + L_y\hat{\boldsymbol{y}} + L_z\hat{\boldsymbol{z}} = \sum_q (-1)^q L_q\hat{\boldsymbol{e}}_{-q}\,, \tag{24}$$

where the sum over q runs over $q=-1,0,+1$.[1] Hence, eqs. (22) and (24) yield

$$\vec{\boldsymbol{L}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar\sqrt{\ell(\ell+1)}\,\sum_q \hat{\boldsymbol{e}}_{-q}\langle \ell\,,\, m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,Y_{\ell\,,\,m+q}(\hat{\boldsymbol{n}})\,.$$

Since the sum is taken over $q=-1,0,1$, we are free to relabel $q\to -q$. Writing $\hat{\boldsymbol{e}}_q = \chi_{1,q}$, we end up with

$$\vec{\boldsymbol{L}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar\sqrt{\ell(\ell+1)}\,\sum_q \langle \ell\,,\, m-q\,;\,1\,,\,q\,|\,\ell\, m\rangle\,Y_{\ell\,,\,m-q}(\hat{\boldsymbol{n}})\chi_{1\,q}\,.$$

Comparing with eq. (4) for $s=1$, it follows that [in the notation of eq. (19)]:

$$\boxed{\vec{\boldsymbol{L}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar\sqrt{\ell(\ell+1)}\,\vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}})} \tag{25}$$

in agreement with eq. (10).

Next, we examine $\hat{\boldsymbol{n}}Y_{\ell m}(\hat{\boldsymbol{n}})$. It is convenient to expand $\hat{\boldsymbol{n}}\equiv\vec{\boldsymbol{x}}/r$ in a spherical basis. Using eq. (23), the following expression is an identity,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta = \sqrt{\frac{4\pi}{3}}\sum_q (-1)^q\,Y_{1q}(\hat{\boldsymbol{n}})\,\hat{\boldsymbol{e}}_{-q}\,. \tag{26}$$

Hence,

$$\hat{\boldsymbol{n}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \sqrt{\frac{4\pi}{3}}\sum_q (-1)^q\,Y_{1q}(\hat{\boldsymbol{n}})Y_{\ell m}(\hat{\boldsymbol{n}})\,\hat{\boldsymbol{e}}_{-q}\,. \tag{27}$$

Using eq. (88) given in Appendix A, it follows that

$$Y_{1q}(\hat{\boldsymbol{n}})Y_{\ell m}(\hat{\boldsymbol{n}}) = \sqrt{\frac{3(2\ell+1)}{4\pi}}\sum_{\ell'}\frac{1}{\sqrt{2\ell'+1}}\,\langle \ell\, m\,;\,1\,q\,|\,\ell'\,,\,m+q\rangle\langle \ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle\,Y_{\ell'\,,\,m+q}(\hat{\boldsymbol{n}})\,, \tag{28}$$

[1]Henceforth, if left unspecified, sums over q will run over $q=-1,0,+1$.

Only two terms, corresponding to $\ell' = \ell \pm 1$, can contribute to the sum over ℓ' since [cf. Table 2]:

$$\langle \ell\, 0\,;\, 1\, 0\,|\, \ell'\, 0\rangle = \begin{cases} \left(\dfrac{\ell+1}{2\ell+1}\right)^{1/2}, & \text{for } \ell' = \ell+1\,, \\[2ex] 0\,, & \text{for } \ell' \neq \ell \pm 1\,, \\[2ex] -\left(\dfrac{\ell}{2\ell+1}\right)^{1/2}, & \text{for } \ell' = \ell - 1\,. \end{cases} \tag{29}$$

Inserting eq. (28) on the right hand side of eq. (27) and employing eq. (29) then yields

$$\begin{aligned} \hat{\boldsymbol{n}} Y_{\ell m}(\hat{\boldsymbol{n}}) = \sum_q (-1)^q\, \hat{\boldsymbol{e}}_{-q} \Biggl\{ \left(\frac{\ell+1}{2\ell+3}\right)^{1/2} \langle \ell\, m\,;\, 1\, q\,|\, \ell+1\,,\, m+q\rangle\, Y_{\ell+1\,,\,m+q}(\hat{\boldsymbol{n}}) \\ - \left(\frac{\ell}{2\ell-1}\right)^{1/2} \langle \ell\, m\,;\, 1\, q\,|\, \ell+1\,,\, m+q\rangle\, Y_{\ell-1\,,\,m+q}(\hat{\boldsymbol{n}}) \Biggr\}. \end{aligned} \tag{30}$$

It is convenient to rewrite eq. (30) with the help of the following two relations, which can be obtained from Table 2,

$$\langle \ell\, m\,;\, 1\, q\,|\, \ell+1\,,\, m+q\rangle = -(-1)^q \left(\frac{2\ell+3}{2\ell+1}\right)^{1/2} \langle \ell+1\,,\, m+q\,;\, 1\,,\, -q\,|\, \ell\, m\rangle\,, \tag{31}$$

$$\langle \ell\, m\,;\, 1\, q\,|\, \ell-1\,,\, m+q\rangle = -(-1)^q \left(\frac{2\ell-1}{2\ell+1}\right)^{1/2} \langle \ell-1\,,\, m+q\,;\, 1\,,\, -q\,|\, \ell\, m\rangle\,. \tag{32}$$

The end result is:

$$\begin{aligned} \hat{\boldsymbol{n}} Y_{\ell m}(\hat{\boldsymbol{n}}) = -\sum_q \hat{\boldsymbol{e}}_{-q} \Biggl\{ \left(\frac{\ell+1}{2\ell+1}\right)^{1/2} \langle \ell+1\,,\, m+q\,;\, 1\,,\, -q\,|\, \ell\, m\rangle\, Y_{\ell+1\,,\,m+q}(\hat{\boldsymbol{n}}) \\ - \left(\frac{\ell}{2\ell+1}\right)^{1/2} \langle \ell-1\,,\, m+q\,;\, 1\,,\, -q\,|\, \ell\, m\rangle\, Y_{\ell-1\,,\,m+q}(\hat{\boldsymbol{n}}) \Biggr\}. \end{aligned} \tag{33}$$

Using eq. (4) with $s = 1$ and $\chi_{1\,q} = \hat{\boldsymbol{e}}_q$ and employing the notation of eq. (19), it follows that

$$\vec{\boldsymbol{Y}}_{\ell\,,\,\ell\pm1\,,\,m}(\hat{\boldsymbol{n}}) = \sum_q \hat{\boldsymbol{e}}_{-q}\, \langle \ell \pm 1\,,\, m+q\,;\, 1\,,\, -q\,|\, \ell\, m\rangle\, Y_{\ell\pm1\,,\,m+q}(\hat{\boldsymbol{n}})\,, \tag{34}$$

after relabeling the summation index by $q \to -q$. Hence, eq. (33) yields

$$\boxed{\hat{\boldsymbol{n}} Y_{\ell m}(\hat{\boldsymbol{n}}) = -\left(\frac{\ell+1}{2\ell+1}\right)^{1/2} \vec{\boldsymbol{Y}}_{\ell\,,\,\ell+1\,,\,m}(\hat{\boldsymbol{n}}) + \left(\frac{\ell}{2\ell+1}\right)^{1/2} \vec{\boldsymbol{Y}}_{\ell\,,\,\ell-1\,,\,m}(\hat{\boldsymbol{n}})} \tag{35}$$

Finally, we examine $r\vec{\boldsymbol{\nabla}} Y_{\ell m}(\hat{\boldsymbol{n}})$. First, we introduce the gradient operator in a spherical basis, $\nabla_q = (\nabla_{+1}\,,\, \nabla_0\,,\, \nabla_{-1})$, where

$$\nabla_{\pm1} = \mp\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right) = \mp\frac{e^{\pm i\phi}}{\sqrt{2}}\left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \pm \frac{i}{r\sin\theta}\frac{\partial}{\partial\phi}\right], \tag{36}$$

$$\nabla_0 = \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\,. \tag{37}$$

We can introduce a formal operator ∇_q on the Hilbert space by defining the coordinate space representation,

$$\langle\vec{\boldsymbol{x}}|\nabla_q|\ell\, m\rangle = \nabla_q Y_{\ell m}(\hat{\boldsymbol{n}})\,.$$

Note that ∇_q is a vector operator. We shall employ the Wigner-Eckart theorem (see, e.g., pp. 240–241 of Ref. [2]), which states that

$$\langle\ell'\, m'|\nabla_q|\ell\, m\rangle = \langle\ell\, m\,;\,1\,q\,|\,\ell'\, m'\rangle\,\langle\ell'\|\nabla\|\ell\rangle\,, \tag{38}$$

where the reduced matrix element $\langle\ell\|\nabla\|\ell'\rangle$ is independent of q, m and m'. To evaluate the reduced matrix element, we consider the case of $q = m = m' = 0$. Then,

$$\langle\ell'\,0|\nabla_0|\ell\,0\rangle = \langle\ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle\,\langle\ell'\|\nabla\|\ell\rangle\,.$$

Thus,

$$\langle\ell'\|\nabla\|\ell\rangle = \frac{\langle\ell'\,0|\nabla_0|\ell\,0\rangle}{\langle\ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle}\,.$$

Inserting this result into eq. (38) yields

$$\langle\ell'\, m'|\nabla_q|\ell\, m\rangle = \frac{\langle\ell\, m\,;\,1\,q\,|\,\ell'\, m'\rangle}{\langle\ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle}\,\langle\ell'\,0|\nabla_0|\ell\,0\rangle\,. \tag{39}$$

We can evaluate $\langle\ell'\,0|\nabla_0|\ell\,0\rangle$ explicitly in the coordinate representation by employing eq. (37),

$$\langle\ell'\,0|\nabla_0|\ell\,0\rangle = -\frac{1}{r}\int d\Omega\, Y^*_{\ell'0}(\hat{\boldsymbol{n}})\sin\theta\,\frac{\partial}{\partial\theta}\,Y_{\ell 0}(\hat{\boldsymbol{n}})\,.$$

Using $Y_{\ell 0}(\hat{\boldsymbol{n}}) = [(2\ell+1)/(4\pi)]^{1/2}\,P_\ell(\cos\theta)$, and substituting $x \equiv \cos\theta$,

$$\langle\ell'\,0|\nabla_0|\ell\,0\rangle = \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2r}\int_{-1}^{1}(1-x^2)P_{\ell'}(x)P'_\ell(x)\,dx\,, \tag{40}$$

where $P'_\ell(x) = dP_\ell(x)/dx$. To evaluate eq. (40), we make use of the recurrence relation,

$$(1-x^2)P'_\ell(x) = \ell P_{\ell-1}(x) - \ell x P_\ell(x)\,,$$

and the orthogonality relation of the Legendre polynomials,

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}\,.$$

It follows that

$$\langle\ell'\,0|\nabla_0|\ell\,0\rangle = \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2r}\left\{\frac{2\ell}{2\ell-1}\,\delta_{\ell',\ell-1} - \ell\int_{-1}^{1} xP_\ell(x)P_{\ell'}(x)\,dx\right\}. \tag{41}$$

To evaluate the remaining integral, we use $x = P_1(x)$ and the result of eq. (91) obtained in Appendix A to write:

$$\int_{-1}^{1} xP_\ell(x)P_{\ell'}(x)\,dx = \int_{-1}^{1} P_1(x)P_\ell(x)P_{\ell'}(x)\,dx = \frac{2}{2\ell'+1}\langle 1\,0\,;\,\ell\,0\,|\,\ell'\,0\rangle^2\,.$$

Using eq. (29), the above integral is equal to

$$\int_{-1}^{1} xP_\ell(x)P_{\ell'}(x)\,dx = \frac{2(\ell+1)}{(2\ell+1)(2\ell+3)}\,\delta_{\ell',\ell+1} + \frac{2\ell}{(2\ell-1)(2\ell+1)}\,\delta_{\ell',\ell-1}\,.$$

Inserting this result back into eq. (41) yields

$$\begin{aligned}\langle \ell'\,0|\nabla_0|\ell\,0\rangle &= \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2r}\left\{\frac{2\ell(\ell+1)}{(2\ell-1)(2\ell+1)}\,\delta_{\ell',\ell-1} - \frac{2\ell(\ell+1)}{(2\ell+1)(2\ell+3)}\,\delta_{\ell',\ell+1}\right\}\\ &= \frac{\ell(\ell+1)}{r\sqrt{2\ell+1}}\left[\frac{1}{\sqrt{2\ell-1}}\,\delta_{\ell',\ell-1} - \frac{1}{\sqrt{2\ell+3}}\,\delta_{\ell',\ell+1}\right].\end{aligned}$$

Using eq. (39), it follows that:

$$\begin{aligned}\langle \ell'\,m'|\nabla_q|\ell\,m\rangle &= \frac{\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle}{\langle \ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle}\,\frac{\ell(\ell+1)}{r\sqrt{2\ell+1}}\left[\frac{1}{\sqrt{2\ell-1}}\,\delta_{\ell',\ell-1} - \frac{1}{\sqrt{2\ell+3}}\,\delta_{\ell',\ell+1}\right]\\ &= -\frac{1}{r}\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle\left[(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\delta_{\ell',\ell-1} + \ell\sqrt{\frac{\ell+1}{2\ell+3}}\,\delta_{\ell',\ell+1}\right], \qquad (42)\end{aligned}$$

after employing eq. (29) to evaluate $\langle \ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle$.

We are now ready to evaluate $r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})$. First, we insert a complete set of states to obtain

$$\begin{aligned}\nabla_q|\ell\,m\rangle &= \sum_{\ell',m'}|\ell'\,m'\rangle\langle \ell'\,m'|\nabla_q|\ell\,m\rangle\\ &= -\frac{1}{r}\sum_{\ell',m'}|\ell'\,m'\rangle\left\{\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle\left[(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\delta_{\ell',\ell-1} + \ell\sqrt{\frac{\ell+1}{2\ell+3}}\,\delta_{\ell',\ell+1}\right]\right\}.\end{aligned} \qquad (43)$$

Note that in the sum over m', only the terms corresponding to $m' = m+q$ survive, due to the presence of the Clebsch-Gordon coefficient $\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle$. Likewise, in the sum over ℓ', only the terms corresponding to $\ell' = \ell \pm 1$ survive. In the coordinate representation, eq. (43) is equivalent to

$$\nabla_q Y_{\ell m}(\hat{\boldsymbol{n}}) = -\frac{1}{r}\sum_{\ell'} Y_{\ell',m+q}(\hat{\boldsymbol{n}})\left\{\langle \ell\,m\,;\,1\,q\,|\,\ell'\,,\,m{+}q\rangle\left[(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\delta_{\ell',\ell-1} + \ell\,\sqrt{\frac{\ell+1}{2\ell+3}}\,\delta_{\ell',\ell+1}\right]\right\}.$$

In analogy with eq. (24), we have

$$\vec{\boldsymbol{\nabla}} = \sum_q (-1)^q\,\hat{\boldsymbol{e}}_{-q}\nabla_q\,.$$

Hence, it follows that

$$\begin{aligned}-r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \sum_q (-1)^q\,\hat{\boldsymbol{e}}_{-q}\bigg\{&(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\langle \ell\,m\,;\,1\,q\,|\,\ell-1\,,\,m+q\rangle Y_{\ell-1,m+q}(\hat{\boldsymbol{n}})\\ &+\ell\,\sqrt{\frac{\ell+1}{2\ell+3}}\,\langle \ell\,m\,;\,1\,q\,|\,\ell+1\,,\,m+q\rangle Y_{\ell+1,m+q}(\hat{\boldsymbol{n}})\bigg\}.\end{aligned}$$

It is convenient to employ eqs. (31) and (32) and rewrite the above result as

$$r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \sum_q \hat{\boldsymbol{e}}_{-q}\Biggl\{(\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\langle \ell-1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\,m\rangle Y_{\ell-1,m+q}(\hat{\boldsymbol{n}}) + \ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\langle \ell+1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\,m\rangle Y_{\ell+1,m+q}(\hat{\boldsymbol{n}})\Biggr\}.$$

Finally, using eq. (34), we end up with

$$\boxed{r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}}) + (\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})} \tag{44}$$

which is known in the literature as the *gradient formula*.

We can now use eqs. (35) and (44) to solve for $\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})$ and $\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})$ in terms of $\hat{\boldsymbol{n}}Y_{\ell m}(\hat{\boldsymbol{n}})$ and $r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})$. Since these are linear equations, they are easily inverted, and we find

$$\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}}) = \frac{1}{\sqrt{(\ell+1)(2\ell+1)}}\left[-(\ell+1)\hat{\boldsymbol{n}} + r\vec{\boldsymbol{\nabla}}\right]Y_{\ell m}(\hat{\boldsymbol{n}})\,, \qquad \text{for } \ell = 0,1,2,3,\ldots, \tag{45}$$

$$\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}}) = \frac{1}{\sqrt{\ell(2\ell+1)}}\left[\ell\hat{\boldsymbol{n}} + r\vec{\boldsymbol{\nabla}}\right]Y_{\ell m}(\hat{\boldsymbol{n}})\,, \qquad \text{for } \ell = 1,2,3,\ldots, \tag{46}$$

which are equivalent to the results of eqs. (14) and (15) previously obtained.[2] In addition, we also have eq. (25), which we can rewrite as

$$\vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}}) = \frac{-ir}{\sqrt{\ell(\ell+1)}}\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\hat{\boldsymbol{n}})\,, \qquad \text{for } \ell = 1,2,3,\ldots,. \tag{47}$$

Thus, we have identified the three linearly independent vector spherical harmonics in terms of differential vector operators acting on $Y_{\ell m}(\hat{\boldsymbol{n}})$. For the special case of $\ell = 0$, only one vector spherical harmonic, $\vec{\boldsymbol{Y}}_{010}(\hat{\boldsymbol{n}}) = (-\hat{\boldsymbol{n}} + r\vec{\boldsymbol{\nabla}})Y_{00}(\hat{\boldsymbol{n}}) = -\hat{\boldsymbol{n}}/\sqrt{4\pi}$, survives.

In the notation of Ref. [13], the three linearly independent normalized vector spherical harmonics obtained above are denoted by

$$\vec{\boldsymbol{X}}_{\ell m} \equiv \vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}})\,, \qquad \vec{\boldsymbol{V}}_{\ell m} \equiv \vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}}) \qquad \vec{\boldsymbol{W}}_{\ell m} \equiv \vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})\,. \tag{48}$$

These vector spherical harmonics satisfy orthonormality relations,

$$\int d\Omega\,\vec{\boldsymbol{X}}^*_{\ell m}\boldsymbol{\cdot}\vec{\boldsymbol{X}}_{\ell' m'} = \int d\Omega\,\vec{\boldsymbol{V}}^*_{\ell m}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\ell' m'} = \int d\Omega\,\vec{\boldsymbol{W}}^*_{\ell m}\boldsymbol{\cdot}\vec{\boldsymbol{W}}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'}\,, \tag{49}$$

for ℓ, $\ell' \geq 1$. Moreover,

$$\int d\Omega\,\vec{\boldsymbol{X}}^*_{\ell m}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\ell' m'} = \int d\Omega\,\vec{\boldsymbol{X}}^*_{\ell m}\boldsymbol{\cdot}\vec{\boldsymbol{W}}_{\ell' m'} = \int d\Omega\,\vec{\boldsymbol{V}}^*_{\ell m}\boldsymbol{\cdot}\vec{\boldsymbol{W}}_{\ell' m'} = 0\,. \tag{50}$$

[2] First, replace ℓ with j in eqs. (45) and (46). Then, in light of eq. (19), $\vec{\boldsymbol{Y}}_{j,j+1,m} = \mathcal{Y}^{j+1,1}_{jm} = \mathcal{Y}^{\ell\,1}_{j=\ell-1,m}$ and $\vec{\boldsymbol{Y}}_{j,j-1,m} = \mathcal{Y}^{j-1,1}_{jm} = \mathcal{Y}^{\ell\,1}_{j=\ell+1,m}$. Using these results, one reproduces eqs. (14) and (15).

Likewise the linearly independent normalized vector spherical harmonics,

$$\left\{\frac{-ir}{\sqrt{\ell(\ell+1)}}\,\boldsymbol{\hat{n}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\boldsymbol{\hat{n}})\,,\quad \boldsymbol{\hat{n}}\,Y_{\ell m}(\boldsymbol{\hat{n}})\,,\quad \frac{r}{\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\boldsymbol{\hat{n}})\right\} \tag{51}$$

satisfy orthonormality relations analogous to those of eqs. (49) and (50).

In the literature, one sometimes encounters another vector spherical harmonic that is defined by $\boldsymbol{\hat{n}}\times\vec{\boldsymbol{X}}_{\ell m}(\boldsymbol{\hat{n}})\equiv\boldsymbol{\hat{n}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\boldsymbol{\hat{n}})/\sqrt{\hbar^2\ell(\ell+1)}$ [for example, see eqs. (80) and (82) in Section 6]. However, $\boldsymbol{\hat{n}}\times\vec{\boldsymbol{X}}_{\ell m}(\boldsymbol{\hat{n}})$ is not independent of the vector spherical harmonics obtained above in light of eq. (17). In particular,

$$\boldsymbol{\hat{n}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\boldsymbol{\hat{n}})=-i\hbar r\,\boldsymbol{\hat{n}}\times(\boldsymbol{\hat{n}}\times\vec{\boldsymbol{\nabla}})Y_{\ell m}(\boldsymbol{\hat{n}})=-i\hbar r\,\left[\boldsymbol{\hat{n}}\frac{\partial}{\partial r}-\vec{\boldsymbol{\nabla}}\right]Y_{\ell m}(\boldsymbol{\hat{n}})=i\hbar r\,\vec{\boldsymbol{\nabla}}Y_{\ell m}(\boldsymbol{\hat{n}})\,. \tag{52}$$

An alternative method for deriving the gradient formula [obtained in eq. (44)] is to evaluate $\boldsymbol{\hat{n}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\boldsymbol{\hat{n}})$ using the same technique employed in the computation of $\boldsymbol{\hat{n}}Y_{\ell m}(\boldsymbol{\hat{n}})$ given in this section. However, this calculation is much more involved and involves a product of four Clebsch-Gordon coefficients. A certain sum involving a product of three Clebsch-Gordon coefficients needs to be performed in closed form. This summation can be done (e.g., see Ref. [11] for the gory details), but the computation is much more involved than the simple analysis presented in this section based on the Wigner-Eckart theorem.

We have seen above that there are numerous choices for a basis set of three mutually orthonormal vector spherical harmonics that satisfy orthonormality relations such as eqs. (49) and (50). Two possible choices are given in eqs. (48) and (51), respectively. More generally, one can choose $\vec{\boldsymbol{X}}_{\ell m}$ as one of the three vector spherical harmonics and choose the other two to be appropriate linear combinations of $\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}$ and $\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}$. For example, consider the quantity $ir\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\boldsymbol{\hat{n}})$. This can be evaluated by employing the following operator identity,

$$i\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}=\hbar r\boldsymbol{\hat{n}}\vec{\boldsymbol{\nabla}}^2-\hbar\vec{\boldsymbol{\nabla}}\left(1+r\frac{\partial}{\partial r}\right)\,. \tag{53}$$

Since $Y_{\ell m}(\boldsymbol{\hat{n}})$ is independent of r, it follows that

$$\frac{i}{\hbar}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\boldsymbol{\hat{n}})=\big[r\boldsymbol{\hat{n}}\vec{\boldsymbol{\nabla}}^2-\vec{\boldsymbol{\nabla}}\big]Y_{\ell m}(\boldsymbol{\hat{n}})\,. \tag{54}$$

Using eq. (3) and

$$\vec{\boldsymbol{\nabla}}^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\vec{\boldsymbol{L}}^2}{\hbar^2r^2}\,, \tag{55}$$

it follows that

$$\frac{i}{\hbar}\,r\,\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\boldsymbol{\hat{n}})=-\ell(\ell+1)\boldsymbol{\hat{n}}Y_{\ell m}(\boldsymbol{\hat{n}})-r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\boldsymbol{\hat{n}})\,. \tag{56}$$

Finally, using the expressions given in eqs. (35) and (44), we end up with

$$\boxed{\frac{i}{\hbar}\,r\,\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\boldsymbol{\hat{n}})=\ell^2\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\boldsymbol{\hat{n}})-(\ell+1)^2\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\boldsymbol{\hat{n}})} \tag{57}$$

Note that the corresponding properly normalized vector spherical harmonic is

$$\vec{\boldsymbol{Z}}_{\ell m}(\hat{\boldsymbol{n}}) \equiv \frac{1}{\sqrt{\ell(\ell+1)(\ell^2+\ell+1)}}\,\frac{i}{\hbar}\, r\,\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\hat{\boldsymbol{n}})\,, \tag{58}$$

which ensures that

$$\int d\Omega\,\vec{\boldsymbol{Z}}^*_{\ell m}\boldsymbol{\cdot}\vec{\boldsymbol{Z}}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'}\,, \tag{59}$$

for ℓ, $\ell' \geq 1$.

6 Multipole electromagnetic radiation fields and the vector spherical harmonics

In this section, we shall set $\hbar = 1$, since $\hbar$ plays no role in what follows. In particular, note that in this convention $\vec{\boldsymbol{L}} = -i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}$, which matches the definition employed in Ref. [14].

Consider the harmonic time-dependent electric and magnetic field vectors,

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \mathrm{Re}\,\big[\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})e^{-i\omega t}\big]\,, \qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) = \mathrm{Re}\,\big[\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})e^{-i\omega t}\big]\,. \tag{60}$$

Following Ref. [14], the electric (ℓ,m)-multipole radiation fields are given by

$$\vec{\boldsymbol{B}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}}) = \frac{a_E(\ell,m)}{\sqrt{\ell(\ell+1)}}\vec{\boldsymbol{L}}\big[h^{(1)}_\ell(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big]\,, \tag{61}$$

$$\vec{\boldsymbol{E}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}}) = \frac{a_E(\ell,m)}{\sqrt{\ell(\ell+1)}}\frac{i}{k}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[h^{(1)}_\ell(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big]\,, \tag{62}$$

and the magnetic (ℓ,m)-multipole radiation fields are given by

$$\vec{\boldsymbol{E}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}}) = \frac{a_M(\ell,m)}{\sqrt{\ell(\ell+1)}}\vec{\boldsymbol{L}}\big[h^{(1)}_\ell(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big]\,, \tag{63}$$

$$\vec{\boldsymbol{B}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}}) = \frac{a_M(\ell,m)}{\sqrt{\ell(\ell+1)}}\left(-\frac{i}{k}\right)\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[h^{(1)}_\ell(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big]\,, \tag{64}$$

where $k \equiv \omega/c$, $r \equiv |\vec{\boldsymbol{x}}|$ is assumed to be much larger than the distance scale that characterizes the current and charge sources (the so-called radiation zone), and $h^{(1)}(kr)$ is a spherical Hankel function that is given by the following closed-form expression:[3]

$$h^{(1)}_\ell(kr) = (-i)^{\ell+1}\frac{e^{ikr}}{kr}\left[1+\sum_{n=1}^{\ell}\frac{(\ell+n)!}{n!(\ell-n)!}\,(-2ikr)^{-n}\right]\,. \tag{65}$$

The coefficients, $a_{E,M}(\ell,m)$, are proportional to the electric and magnetic multipole tensors, respectively. In particular (e.g., see the 2nd edition of Ref. [14]),

$$a_E(\ell,m) = -\left(\frac{\ell+1}{\ell}\right)^{1/2}\frac{4\pi i k^{\ell+2}}{(2\ell+1)!!}\,Q_{\ell m}\,, \tag{66}$$

$$a_M(\ell,m) = \left(\frac{\ell+1}{\ell}\right)^{1/2}\frac{4\pi i k^{\ell+2}}{(2\ell+1)!!}\,M_{\ell m}\,. \tag{67}$$

[3]See formula 10.49.6 of Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, UK, 2010).

Under the assumption of harmonic sources $\rho(\vec{\boldsymbol{x}},t) = \text{Re}\,\left[\rho(\vec{\boldsymbol{x}})e^{-i\omega t}\right]$ and $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \text{Re}\,\left[\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})e^{-i\omega t}\right]$, the electric and magnetic multipole spherical tensors (in gaussian units) are:

$$Q_{\ell m} \equiv \int d^3x\, r^\ell Y^*_{\ell m}(\hat{\boldsymbol{n}})\rho(\vec{\boldsymbol{x}})\,, \tag{68}$$

$$M_{\ell m} \equiv \frac{1}{c(\ell+1)}\int d^3x\,\left[\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\right]\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\big(r^\ell Y^*_{\ell m}(\hat{\boldsymbol{n}})\big)\,. \tag{69}$$

One important property of $\vec{\boldsymbol{B}}^{(E,M)}_{\ell m}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{E}}^{(E,M)}_{\ell m}(\vec{\boldsymbol{x}})$ is that $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}^{(E,M)}_{\ell m} = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}^{(E,M)}_{\ell m} = 0$. These results are automatically satisfied due to the operator identities:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{L}} = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}) = 0\,. \tag{70}$$

In light of eqs. (9) and (11), it follows that

$$\frac{1}{\sqrt{\ell(\ell+1)}}\vec{\boldsymbol{L}}\big[h^{(1)}_\ell(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big] = h^{(1)}_\ell(kr)\vec{\boldsymbol{X}}_{\ell m}(\hat{\boldsymbol{n}})\,. \tag{71}$$

Thus, $\vec{\boldsymbol{B}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{E}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}})$ are proportional to the vector spherical harmonic $\vec{\boldsymbol{X}}_{\ell m}(\hat{\boldsymbol{n}})$. In contrast, $\vec{\boldsymbol{E}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{B}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}})$ are proportional to $-i\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[h^{(1)}_\ell(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big]$. The latter can be evaluated by employing eq. (98) of Appendix B, which yields:

$$\begin{aligned}-i\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[f(r)Y_{\ell m}(\hat{\boldsymbol{n}})\big] = \ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}-\frac{\ell}{r}\right)f(r)\\ +(\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}+\frac{\ell+1}{r}\right)f(r)\,,\end{aligned} \tag{72}$$

after noting that $f(r)\vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}}) = \vec{\boldsymbol{L}}\big[f(r)Y_{\ell m}(\hat{\boldsymbol{n}})\big]/\sqrt{\ell(\ell+1)}$ in light of eq. (9).[4] Next, we make use of the following two derivative formulae that are satisfied by the spherical Hankel functions:

$$\left(\frac{d}{dr}-\frac{\ell}{r}\right)h^{(1)}_\ell(kr) = -kh^{(1)}_{\ell+1}(kr)\,,\qquad \left(\frac{d}{dr}+\frac{\ell+1}{r}\right)h^{(1)}_\ell(kr) = kh^{(1)}_{\ell-1}(kr)\,,\qquad \text{for } \ell\geq 1. \tag{73}$$

It then follows that

$$\frac{i}{k}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[h^{(1)}_\ell(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big] = \ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})h^{(1)}_{\ell+1}(kr) - (\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})h^{(1)}_{\ell-1}(kr)\,. \tag{74}$$

Thus, $\vec{\boldsymbol{E}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{B}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}})$ are linear combinations of the vector spherical harmonics $\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})$ and $\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})$. Note that the divergence of the right-hand side of eq. (74) must vanish, which provides a useful check of this result (see Appendix B for further details).

In the radiation zone, we can employ the asymptotic form for the spherical Hankel function, which is given by the first term on the right-hand side of eq. (65),

$$h^{(1)}_{\ell-1}(kr) \simeq -h^{(1)}_{\ell+1}(kr) \simeq (-i)^\ell\,\frac{e^{ikr}}{kr}\,. \tag{75}$$

[4]If we set $f(r)=1$ in eq. (72), we recover the result previously obtained in eq. (57) [after setting $\hbar = 1$].

Hence,

$$-i\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[h_\ell^{(1)}(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big] \simeq (-i)^\ell\,\frac{e^{ikr}}{r}\left\{\ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})+(\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})\right\}. \tag{76}$$

Using eqs. (45) and (46), we end up with

$$-i\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[h_\ell^{(1)}(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big] \simeq (-i)^\ell\,e^{ikr}\,\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})\,. \tag{77}$$

A more useful form is obtained by employing eq. (52). We then obtain

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}\big[h_\ell^{(1)}(kr)Y_{\ell m}(\hat{\boldsymbol{n}})\big] \simeq (-i)^\ell\,\frac{e^{ikr}}{r}\hat{\boldsymbol{n}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\hat{\boldsymbol{n}})\,. \tag{78}$$

In conclusion, we have succeeded in expressing the asymptotic forms for the multipole radiation fields in terms of the vector spherical harmonics. In particular, the electric (ℓ,m)-multipole radiation fields are given by

$$\vec{\boldsymbol{B}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}}) \simeq (-i)^{\ell+1}a_E(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{X}}_{\ell m}(\hat{\boldsymbol{n}})\,, \tag{79}$$

$$\vec{\boldsymbol{E}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}}) \simeq -(-i)^{\ell+1}a_E(\ell,m)\frac{e^{ikr}}{kr}\hat{\boldsymbol{n}}\times\vec{\boldsymbol{X}}_{\ell m}(\hat{\boldsymbol{n}})\,, \tag{80}$$

and the magnetic (ℓ,m)-multipole radiation fields are given by

$$\vec{\boldsymbol{E}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}}) \simeq (-i)^{\ell+1}a_M(\ell,m)\frac{e^{ikr}}{kr}\vec{\boldsymbol{X}}_{\ell m}(\hat{\boldsymbol{n}})\,, \tag{81}$$

$$\vec{\boldsymbol{B}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}}) \simeq (-i)^{\ell+1}a_M(\ell,m)\frac{e^{ikr}}{kr}\hat{\boldsymbol{n}}\times\vec{\boldsymbol{X}}_{\ell m}(\hat{\boldsymbol{n}})\,. \tag{82}$$

Note that the asymptotic forms of the multipole radiation fields satisfy:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}}) = 0\,, \tag{83}$$

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}^{(E)}_{\ell m}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}^{(M)}_{\ell m}(\vec{\boldsymbol{x}}) = \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{84}$$

where we have made use of the operator identities

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{L}} = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{L}} = \hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{L}}) = 0\,, \tag{85}$$

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{L}}) = (\vec{\boldsymbol{\nabla}}\times\hat{\boldsymbol{n}})\boldsymbol{\cdot}\vec{\boldsymbol{L}} - \hat{\boldsymbol{n}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}}) = -\frac{i}{r}\vec{\boldsymbol{L}}^2\,, \tag{86}$$

and $\vec{\boldsymbol{\nabla}}\times\hat{\boldsymbol{n}}=0$ has been used in the last step above. Of course, $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}^{(E,M)}_{\ell m} = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}^{(E,M)}_{\ell m} = 0$ must be exactly satisfied. Indeed, the $\mathcal{O}(r^{-2})$ terms in eq. (84) will be canceled when higher order terms in the r^{-1} expansion of the radiation fields are included [i.e., when eq. (65) is used instead of the leading $\mathcal{O}(r^{-1})$ term given by eq. (75)].

Appendix A: An integral of a product of three spherical harmonics

In this Appendix, we state a number of results that are derived on pp. 231 of Ref. [2]. The product of two spherical harmonics can be obtained via an important expansion known as the Clebsch-Gordon series,

$$Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})\, Y_{\ell_2 m_2}(\hat{\boldsymbol{n}}) = \sum_{\ell,m} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell_1\, m_1\,;\, \ell_2\, m_2\,|\,\ell\, m\rangle \langle \ell_1\, 0\,;\, \ell_2\, 0\,|\,\ell\, 0\rangle\, Y_{\ell m}(\hat{\boldsymbol{n}}) . \tag{87}$$

The sum over m can be performed using eq. (2). Only one term survives (corresponding to $m = m_1 + m_2$),

$$Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})\, Y_{\ell_2 m_2}(\hat{\boldsymbol{n}}) = \sum_{\ell} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell_1\, m_1\,;\, \ell_2\, m_2\,|\,\ell\;\, m_1{+}m_2\rangle \langle \ell_1\, 0\,;\, \ell_2\, 0\,|\,\ell\, 0\rangle\, Y_{\ell,\, m_1+m_2}(\hat{\boldsymbol{n}}) . \tag{88}$$

Note that $\langle \ell_1\, m_1\,;\, \ell_2\, m_2\,|\,\ell\;\, m_1+m_2\rangle = 0$ unless the two conditions, $|\ell_1-\ell_2| \leq \ell \leq \ell_1+\ell_2$ and $|m_1+m_2| \leq \ell$ are both satisfied. This is simply a consequence of the rules for the addition of angular momentum in quantum mechanics. Consequently, the sum over ℓ in eq. (88) can be taken over the range of integer values that satisfy: $\max\bigl\{|\ell_1-\ell_2|\,,\, m_1+m_2\bigr\} \leq \ell \leq \ell_1+\ell_2$.

If we multiply eq. (87) by $Y^*_{\ell_3 m_3}(\hat{\boldsymbol{n}})$ and then integrate over the solid angle using the orthogonality of the spherical harmonics,

$$\int Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi)\, d\Omega = \delta_{\ell\ell'}\, \delta_{mm'}\,, \tag{89}$$

one easily obtains the integral of a product of three spherical harmonics,

$$\int Y_{\ell_1 m_1}(\theta,\phi)\, Y_{\ell_2 m_2}(\theta,\phi)\, Y^*_{\ell_3 m_3}(\theta,\phi)\, d\Omega = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}}\, \langle \ell_1\, m_1\,;\, \ell_2\, m_2\,|\,\ell_3\, m_3\rangle\, \langle \ell_1\, 0\,;\, \ell_2\, 0\,|\,\ell_3\, 0\rangle\,. \tag{90}$$

Using $Y_{\ell 0}(\hat{\boldsymbol{n}}) = [(2\ell+1)/(4\pi)]^{1/2}\, P_\ell(\cos\theta)$, the following special case of eq. (90) is then obtained,

$$\int_{-1}^{1} P_{\ell_1}(x) P_{\ell_2}(x) P_{\ell_3}(x)\, dx = \frac{2}{2\ell_3+1}\, \langle \ell_1\, 0\,;\, \ell_2\, 0\,|\,\ell_3\, 0\rangle^2\,. \tag{91}$$

Appendix B: Differential operations

The following relations, which can be found in Refs. [3, 8, 12], are often useful. In the formulae below, $f(r)$ denotes an arbitrary function of $r \equiv |\vec{\boldsymbol{x}}|$ and $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$. First, we provide the gradient relation,

$$\vec{\boldsymbol{\nabla}}\bigl[f(r) Y_{\ell m}(\hat{\boldsymbol{n}})\bigr] = \sqrt{\frac{\ell}{2\ell+1}}\, \vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}}) \left(\frac{d}{dr} + \frac{\ell+1}{r}\right) f(r) - \sqrt{\frac{\ell+1}{2\ell+1}}\, \vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}}) \left(\frac{d}{dr} - \frac{\ell}{r}\right) f(r)\,. \tag{92}$$

Note that if one sets $f(r) = 1$ then the above formula reduces to eq. (44).

Next, we present the divergence relations:

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[f(r)\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})\big] = -\sqrt{\frac{\ell+1}{2\ell+1}}\,Y_{\ell m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}+\frac{\ell+2}{r}\right)f(r)\,, \tag{93}$$

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[f(r)\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})\big] = \sqrt{\frac{\ell}{2\ell+1}}\,Y_{\ell m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}-\frac{\ell-1}{r}\right)f(r)\,, \tag{94}$$

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[f(r)\vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}})\big] = 0\,. \tag{95}$$

Finally, we exhibit the curl relations:

$$\vec{\boldsymbol{\nabla}}\times\big[f(r)\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})\big] = i\sqrt{\frac{\ell}{2\ell+1}}\,Y_{\ell m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}+\frac{\ell+2}{r}\right)f(r)\,, \tag{96}$$

$$\vec{\boldsymbol{\nabla}}\times\big[f(r)\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})\big] = i\sqrt{\frac{\ell+1}{2\ell+1}}\,Y_{\ell m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}-\frac{\ell-1}{r}\right)f(r)\,, \tag{97}$$

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\times\big[f(r)\vec{\boldsymbol{Y}}_{\ell\ell m}(\hat{\boldsymbol{n}})\big] &= i\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}-\frac{\ell}{r}\right)f(r)\\ &\quad +i\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})\left(\frac{d}{dr}+\frac{\ell+1}{r}\right)f(r)\,.\end{aligned} \tag{98}$$

As an example, since $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{L}})=0$, it follows that the divergence of the right-hand side of eq. (74) must also vanish. We can check this by employing eqs. (93) and (94):

$$\begin{aligned}&\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\left\{\ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})+(\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})\right\}\\ &\qquad = -\frac{\ell(\ell+1)}{2\ell+1}Y_{\ell m}(\hat{\boldsymbol{n}})\left\{\left(\frac{d}{dr}+\frac{\ell+2}{r}\right)h^{(1)}_{\ell+1}(kr)-\left(\frac{d}{dr}-\frac{\ell-1}{r}\right)h^{(1)}_{\ell-1}(kr)\right\}\\ &\qquad = \frac{k\ell(\ell+1)}{2\ell+1}Y_{\ell m}(\hat{\boldsymbol{n}})\big[h^{(1)}_\ell(kr)-h^{(1)}_\ell(kr)\big]=0\,,\end{aligned} \tag{99}$$

after making use of eq. (73).

Bibliography

The basics of angular momentum theory, Clebsch-Gordon coefficients, spherical tensors and the Wigner-Eckart theorem are treated in most quantum mechanics textbooks. Pedagogical treatments can be found in the following two well-known textbooks:

1. R. Shankar, *Principles of Quantum Mechanics*, 2nd Edition (Springer Science+Business Media, LLC, New York, NY, 1980).

2. J.J. Sakurai and Jim J. Napolitano, *Modern Quantum Mechanics*, 3rd edition (Cambridge University Press, Cambridge, UK, 2021).

An excellent treatment of the angular momentum theory in quantum mechanics that includes a detailed discussion of the vector spherical harmonics can be found in:

3. Guangjiong Ni and Suqing Chen, *Advanced Quantum Mechanics* (Rinton Press, Inc., Princeton NJ, 2002).

Further details can be found in the following more specialized textbooks. In particular, Ref. [8] below is especially comprehensive in its treatment of the tensor spherical harmonics.

4. M.E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, NY, 1957).

5. A.R. Edmonds, *Angular Momentum in Quantum Mechanics*, 2nd Edition (Princeton University Press, Princeton, NJ, 1960).

6. D.M. Brink and G.R. Sachler, *Angular Momentum*, 2nd Edition (Oxford University Press, Oxford, UK, 1968).

7. L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics: Theory and Application* (Cambridge University Press, Cambridge, UK, 1985).

8. D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific Publishing Co., Singapore, 1988).

9. William J. Thompson, *Angular Momentum: An Illustrated Guide to Rotational Symmetries for Physical Systems* (John Wiley & Sons, Inc., New York, NY, 1994).

10. Masud Chaichian and Rolf Hagedorn, *Symmetries in Quantum Mechanics: From Angular Momentum to Supersymmetry* (Institute of Physics Publishing, Ltd, Bristol, UK, 1998).

11. V. Devanathan, *Angular Momentum Techniques in Quantum Mechanics* (Kluwer Academic Publishers, New York, NY, 2002).

A very nice summary of the properties of the vector spherical harmonics can be found in

12. George B. Arfken, Hans J. Weber and Frank E. Harris, *Mathematical Methods for Physicists—A Comprehensive Guide*, 7th Edition (Academic Press, Waltham, MA, 2013). See pp. 809–814.

13. E.L. Hill, *The Theory of Vector Spherical Harmonics*, American Journal of Physics **22**, 211–214 (1954).

Applications of vector spherical harmonics in the theory of electromagnetic radiation can be found in

14. J .D. Jackson, *Classical Electrodynamics*, 3rd Edition (John Wiley & Sons, Inc., New York, 1999).

15. Emil Jan Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, Inc., New York, 1981).

16. John M. Blatt and Victor F. Weisskopf, *Theoretical Nuclear Physics* (Springer-Verlag, New York, 1979).